A LITTLE BIT NASTY, SOME OF THE TIME: MIXED STRATEGY EQUILIBRIA IN POLITICAL CAMPAIGNS WITH CONTINUOUS NEGATIVITY

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Reputation interacts with electoral outcomes in both directions: a candidate with a higher reputation is more likely to win, and winning directly raises reputation. In this context, we assume that negative campaigning is a continuous variable that increases a candidate's chances of winning by lowering the opponent's reputation. However, it also lowers the candidate's own reputation, either due to voters' dislike of candidates who go negative, or because of the opportunity cost of forgoing positive ads. The candidate who ends the campaign with the higher reputation wins the election. For candidates who seek to maximize their post campaign reputations, we derive necessary conditions for mixed strategy equilibria. These conditions are not always compatible, so equilibria do not always exist. For a linear specification, we specify when an equilibrium exists and derive properties when it does. When no continuous equilibrium exists, we consider equilibria for discrete approximations to the game.

1. Introduction

The reputations of political candidates, as well as their policy positions, are important in determining electoral outcomes. As argued by Ledyard (1986), however, capturing the intuitive idea of reputation is not simple. Many models tie reputation to policy positions either prospectively (how committed are the candidates to their current promises?) or retrospectively (did they carry out past promises?). Not incorporated in these models is how campaign activities affect candidate reputations, and the value candidates might place on reputation independent of its effect on electoral outcomes. In this paper, we develop a model that focuses entirely on reputation, independent of policy positions the candidates may take. Reputation can be based on factors such as character or competence, rather than issue positions as in most models of electoral equilibrium.

One of the most important factors affecting candidate reputations through campaigning is negative advertising, which is widely perceived to be an important aspect of political contests and to be growing in significance. Despite this, only a small number of models have been developed that analyze the decision by candidates to run positive or negative campaigns. These include Skaperdas and Grofman (1995), Harrington and Hess (1996), and Fletcher and Slutsky (2010), each of which focuses on a different aspect of negative campaigning.

Skaperdas and Grofman (1995) assume that there are three types of voters: undecideds and those initially favoring each of the two candidates. Undecided individuals shift to supporting one or the other of the two candidates with the division based upon how much positive campaigning each candidate does. Negative campaigning has no direct effect on undecideds but is undertaken to shift some of the opponent's supporters into being undecided. Due to a boomerang effect, it also loses some of the candidate's own supporters to the undecided status. Skaperdas and Grofman assume that these effects vary continuously with the actions taken, that there are diminishing returns to positive campaigning, and that candidates seek to maximize their net votes. They show that there is a pure strategy Nash equilibrium under these assumptions and derive properties of that equilibrium. Among these properties are that the frontrunner (the candidate who initially has more supporters) does more positive and less negative campaigning than the opponent, and that a candidate does more negative campaigning if the opponent has more initial supporters.

Harrington and Hess (1996) impose more structure on individual preferences and are more specific on what campaigning seeks to do. They assume that voters have preferences over both the issue positions of the candidates and over candidate attributes. The voters differ in their ideal points in issue space but have common perceptions of where each of the candidates is located in that issue space. Candidate attributes are valence issues such as character or competence, which all voters value identically. The purpose of campaigning is to shift voters' perceptions of where candidates are in the issue space. Positive campaigning seeks to move the perceptions of the candidate's own position, whereas negative campaigning seeks to move perceptions of the opponent's issue position. Campaigning of either type moves the issue positions continuously and with non-increasing returns. Each candidate chooses a mix of campaigning to maximize his vote share. Harrington and Hess provide conditions for the existence of a pure strategy Nash equilibrium, and show that the candidate who is stronger with respect to valence issues runs a more positive campaign than the opponent.

While candidates do campaign about policy issues as in Harrington and Hess, they also campaign about valence traits, for example, attacking an opponent's honesty or competence in a negative campaign. Fletcher and Slutsky (2010) consider positive and negative campaigns in a context with only valence issues. They assume that candidates have reputations entering the campaign and that a positive campaign raises the candidate's own reputation, while a negative campaign lowers the opponent's reputation. Unlike both Skaperdas - Grofman and Harrington - Hess, Fletcher and Slutsky assume that the decision of whether to run a positive or negative campaign is discrete and not continuous. That is, a campaign is either entirely positive or entirely negative, with nothing in between.

In addition, they assume that the winner is the candidate with the higher post-campaign reputation. Thus, again in contrast to the previous studies, they assume that campaign activities have a discontinuous effect on outcomes given that the candidates maximize the sum of their post-campaign reputations and a reputation bonus from winning. The combination of the discreteness and the discontinuous effects yields Nash equilibria in mixed strategies for a range of parameter values. They show that the candidates' expected payoffs are not monotonically increasing in pre-campaign reputations. When the campaign game is embedded as the second stage in a two stage model where pre-campaign reputations are chosen in the first stage, this non-monotonicity means that candidates will often not choose the maximum possible reputation in the first stage.

This paper considers a model similar to Fletcher and Slutsky (2010) but assumes that the decisions of how negative a campaign to run are continuous instead of being all or nothing. In some ways, this is clearly more realistic, as political campaigns are typically not all positive or all negative. Even with a continuous choice variable, the discontinuity in the election outcome as a function of the candidates' relative reputations implies that pure strategy Nash equilibria will still not exist for a wide range of parameters. This arises for similar reasons to the non-existence of pure strategy equilibria in a Bertrand pricing game where firms benefit by just undercutting their competitor's price. Here, a candidate wants to be just negative enough given the opponent's strategy that that there is not a tie in post-campaign reputations.

In Theorem 1, for a very general specification within the reputation framework, we derive necessary conditions on the equilibrium cumulative distribution functions over the degree of negativity. If there is an equilibrium, the frontrunner's cdf has an atom at zero negativity and then increases continuously up to a level strictly less than complete negativity. The trailing candidate's cdf may or may not have an atom at 0, and then is strictly increasing on an interval strictly between 0 and 1. Additional restrictions must hold for cdfs satisfying these necessary conditions to be an equilibrium. These additional restrictions may not be satisfied, in which case the game has no equilibrium, in pure or mixed strategies.

To gain more insight into when an equilibrium does and does not exist, we construct a specific, relatively linear example, explicitly derive the cdfs, and determine explicit restrictions on the parameters for an equilibrium to exist. Nonexistence can arise for a wide range of parameter values, including the important case when the bonus to reputation is large relative to the campaign effects on reputation, so that the major motivation of candidates is to win the election. To explore what might happen when no continuous equilibrium exists, we numerically solve for Nash equilibria in discrete approximations of the continuous game where the degree of negativity is a discrete variable. The main qualitative differences are that there is a second atom in the frontrunner's distribution at the top of the support, and that the trailing candidate's interval on which there is positive probability ends at complete negativity.

When an equilibrium exists in the continuous game, we show that the frontrunner's campaign is always more positive on average than that of the challenger. This is consistent with results in Skaperdas-Grofman and Harrington-Hess. However, when there is no equilibrium in the continuous game, in the discrete approximation game, the frontrunner can be more negative on average than the trailing candidate. An implication of this is that by running the more negative campaign, the initially trailing candidate can have a significant probability of winning. In fact, under equilibrium strategies, the trailing candidate can win with greater probability than the frontrunner.

We also derive some comparative statics results when a continuous equilibrium exists, many of which appear to be counterintuitive. For example, there are circumstances in which an increase in the size of the reputation bonus from winning leads to a candidate being less negative on average rather than being more negative. An increase in the effectiveness of positive campaigning can in some circumstances lead to more negative campaigning on average, while an increase in the effectiveness of negative campaigning sometimes induces the trailing candidate to be more positive. Finally, we consider the effect of making a candidate initially stronger, either due to having a higher initial reputation or due to having more resources, on that candidate's expected payoff from the game and on the opponent's expected payoff. Consistent with Fletcher and

Slutsky (2010), initially strengthening a candidate can make that candidate worse off and the opponent better off.

The model is specified in Section 2. Pure strategy equilibria are considered in Section 3, and the characterization of mixed strategy equilibria is presented in Section 4. An explicit example is derived in Section 5. Section 6 describes discrete approximations of this explicit game, and Section 7 discusses some of its equilibrium properties. Conclusions are given in Section 8, and all proofs appear in an Appendix.

2. The model

Consider an election between candidates F and T who enter the campaign with initial reputations X_F and X_T, which are based upon exogenous traits or pre-campaign actions. The candidates are labeled so that $X_F \geq X_T$, with F the initial frontrunner and T the initially trailing candidate. The candidates affect these reputations by the campaigns they run, which can vary in the degree to which they are positive or negative in orientation. Positive campaigning raises the reputation of the candidate running the campaign, while negative campaigning lowers the reputation of the opponent. Let f and t be the fractions of the campaigns of F and T, respectively, that are negative. The effects of the campaigns on the candidates' reputations are given by functions $\theta^I(i, j)$ where i, j = f, t, $i \neq j$, so that the post-campaign reputations are $X_F + \theta^F(f, t)$ for the frontrunner and $X_T + \theta^T(t, f)$ for the trailing candidate. The functions θ^F and θ^T can differ from each other because candidates or campaigns differ in their effectiveness or in their resources. [1]

Assume that these functions are continuously differentiable in i and j and that:

[1] In a more general model, they could also differ because of differences in the types of voters supporting different candidates. Here, for simplicity, all voters are assumed identical.

$$\partial\theta^J/\partial i < \partial\theta^I/\partial i < 0 \quad (1)$$

Running a more negative campaign lowers the candidate's own reputation --- that is, $\partial\theta^I/\partial i < 0$ holds --- either because of the opportunity cost of reducing the positive campaigning that would have raised the candidate's own reputation, or because there is a "boomerang" effect where voters feel less positive about candidates who attack their opponents. That this effect is smaller in magnitude than the reduction in the opponent's reputation --- that is, $\partial\theta^J/\partial i$ --- means that negative campaigning is effective on the margin in raising the relative reputation of the candidate doing it.

In addition, we make the following boundary assumptions about $\theta^I(i, j)$:

$$\theta^I(0, 0) > 0, \ \ \theta^I(1, 1) < \theta^I(0, 1) < 0, \ \ \theta^I(0, 0) > \theta^I(1, 0) > \theta^I(0, 1) \quad (2)$$

When both candidates run entirely positive campaigns, each candidate's reputation is greater than its pre-campaign level ($\theta^I(0, 0) > 0$). A candidate who runs an entirely negative campaign is successful in lowering the opponent's reputation whether the opponent's campaign is entirely positive or entirely negative ($\theta^I(0, 1) < 0$ and $\theta^I(1, 1) < 0$). The effect on a candidate's own reputation of running an entirely negative campaign against the opponent's entirely positive is ambiguous; $\theta^I(1, 0)$ could be positive or negative. The presence of a strong boomerang effect would tend to make this term negative.

The outcome of the election depends on the relative post-campaign reputations of the candidates $\Delta(f, t) = X_F + \theta^F(f, t) - X_T - \theta^T(t, f)$, where candidate F wins if $\Delta > 0$, T wins if $\Delta < 0$, and the candidates have an equal chance of winning if $\Delta = 0$. This is summarized by a function $h(\Delta)$ giving the probability that F wins, where $h(\Delta) = 1$ if $\Delta > 0$, $h(\Delta) = 0$ if $\Delta < 0$, and $h(0) = ½$. The candidate who wins the election receives a strictly positive bonus to reputation, B_F or B_T. The candidates seek to maximize their expected post-election reputations $U^F(f, t) = X_F + \theta^F(f, t) + h(\Delta)B_F$ and $U^T(t, f) = X_T + \theta^T(t, f) + (1 - h(\Delta))B_T$.

These preferences incorporate different objectives: winning the election, and maximizing reputation separate from the outcome of the election. When B_i is large relative to the other parameters, candidate i's predominant concern is winning the election. Fletcher and Slutsky (2010) call such candidates Lombardians, after the coach's famous precept that "winning isn't the important thing, it's the only thing." An alternative is B_i small relative to the other parameters, which would be Fletcher and Slutsky's "gentleman politician" who lives by the Grantland Rice adage "it's not whether you win or lose, it's how you play the game."

The public choice literature commonly treats maximizing expected vote share as an alternative to maximizing the probability of winning.[2] Reputation separate from winning might seem to connect more to expected vote maximization: even a losing candidate may want to leave the election with a higher reputation for future activities, which could be achieved from a better performance at the ballot box. In this model,

[2] Some of the papers analyzing the relation between maximizing vote share and the probability of winning the election are Aranson, Hinich and Ordeshook (1974), Ledyard (1984), Snyder (1989), Duggan (2000), and Patty (2005).

though, more negative campaigning would not only raise the probability of winning, but also vote share. However, it would lower the candidate's reputation because a negative ad lowers reputation relative to a positive ad.

We assume that there exist levels of f and t such that $\Delta(f, t) = 0$, as shown in Figure 1. This picture will arise if $\Delta(0, 0)$ is positive with F winning when both candidates run completely positive campaigns, if $\Delta(0, 1)$ is negative so that T wins when running a completely negative campaign against F's completely positive one, and if $\Delta(1, 1)$ is positive with F again winning if both run completely negative campaigns. Sufficient conditions for this are that $X_F - X_T$ is positive but not very large, and that θ^F and θ^T are sufficiently responsive to f and t and are reasonably similar in their values. Given monotonicity of θ^I in f and t, the values of f and t for which $\Delta(f, t) = 0$ form a thin curve between points (0, t*) and (f*, 1). Denote the locus of such points as $i = m^I(j)$. That is, $\Delta(m^F(t), t) = 0$ for $t^* \leq t \leq 1$ or, equivalently, $\Delta(f, m^T(f)) = 0$ for $0 \leq f \leq f^*$.

To analyze the game, it is useful to convert it to the mixed extension in terms of the cumulative distribution functions of the players' mixed strategies. Let $\Gamma^I(i)$ be the cumulative distribution function for each player's mixed strategy. As a cdf, Γ^I is right-hand continuous with at most a countable number of discontinuities or atoms. At any point at which there is an atom, denote its magnitude as $\gamma^I(i)$. The expected payoff to candidate I from using any strategy i against candidate J's mixed strategy would be:

$$EU^I(i, \Gamma^J) = X_I + E_j[\theta^I(i, j)] + B_I[\Gamma^J(m^J(i)) - \gamma^J(m^J(i))/2] \quad (3)$$

The expression multiplying B_I is the probability that candidate I wins the election. The γ^J term exists for the cases where J has an atom at $m^J(i)$, creating some probability of a tie, so that candidate I wins the election only half the time.

A Nash equilibrium is a pair of cdfs such that neither candidate, given the other's cdf, can increase his own expected utility by changing his own cdf. In the next two sections, we derive some necessary conditions for such a Nash equilibrium.

3. Pure strategy equilibria

There are only two possible pure strategy equilibria in this model: one where both candidates run entirely positive campaigns, and one where both candidates run entirely negative campaigns. This is shown in Lemma 1.

Lemma 1: The only possible pure strategy equilibria are (a) f = t = 0 when $\Delta(0,0) \neq 0$ and (b) f = t = 1 when $\Delta(1,1) = 0$.

For any other strategy pair, a candidate can either reduce his negativity by a small amount and increase his reputation without decreasing his probability of winning the election, or increase his negativity by a small amount, discontinuously increasing his chance of winning. Note that the logic here is similar to that in a Bertrand model, where in some circumstances a firm may decrease price by a small amount to gain a large increase in market share, while in other circumstances it can increase its market price toward the monopoly price without losing market share.

Reasonable additional parameter restrictions rule out these two pure strategy equilibria. The equilibrium at (1, 1) is ruled out by the assumption that $\Delta(1, 1) > 0$ which occurs when the following holds:

$$X_F - X_T + \theta^F(1, 1) - \theta^T(1, 1) > 0 \qquad (4)$$

That is, both running completely negative campaigns cannot be an equilibrium if the initial reputation advantage of F over T is not overcome by the negative campaigning. [3] If we assume $\Delta(0, 0) > 0$ and $\Delta(0, 1) < 0$ as in Figure 1, then the (0, 0) equilibrium is ruled out if the winning bonus for T is sufficiently large relative to the direct reputation effect of running a negative campaign:

$$B_T > \theta^T(0, 0) - \theta^T(1, 0) \qquad (5)$$

Starting from (0, 0), candidate T loses the election but would instead win it by running a sufficiently negative campaign (at least t*). The gain from doing this would be the extra winning bonus B_T, while the reputation loss from running a more negative campaign would be $\theta^T(0, 0) - \theta^T(t^*, 0) < \theta^T(0, 0) - \theta^T(1, 0)$. Hence, under condition (5), regardless of the exact value of t*, candidate T would gain by raising t from 0 to above t*, ruling out the possible equilibrium in Lemma 1(a).

[3] This would also be ruled out if $\Delta(1, 1) < 0$ or if $\Delta(1, 1) = 0$ and $\theta^I(0, 1) - \theta^I(1, 1) > ½B_I$ for I = F or T. If $\Delta(1, 1) = 0$ does hold, then each candidate's post-election reputation would be $X_I + \theta^I(1, 1) + ½B_I$. Either candidate could concede the election but gain in reputation by running a completely positive campaign. The post-election reputation of that candidate would then be $X_I + \theta^I(0, 1)$. When $\theta^I(0, 1) - \theta^I(1, 1) > ½B_I$, so that the direct reputation gain is more important to the candidate than the loss from never winning the election and candidate I would gain by deviating from (1, 1). We assume (4) since the other cases would yield similar results.

Assuming both (4) and (5) rules out all pure strategy equilibria, leaving only non-degenerate mixed strategies as possible equilibria. These are considered in the next section.

4. Properties of equilibrium mixed strategies

We have shown that the circumstances under which pure strategy equilibria can occur are limited. In fact, there are also circumstances in which this game has no equilibria, even in mixed strategies. This section gives insight into when mixed strategy equilibria do exist through the central result of the paper, a characterization of the mixed strategy equilibria that may occur in the campaign game. This characterization, which is specified in Theorem 1, is developed in a series of lemmas, each of which gives an important property that must hold in any mixed strategy equilibrium.

First, in Lemma 2, we show that the frontrunner can put no probability weight on very negative campaigns, while the trailing candidate can put no probability weight on very positive campaigns, except for possibly having a probability mass at an entirely positive campaign.

Lemma 2: $\Gamma^F(1) = \Gamma^F(f^*) = 1$ and $\Gamma^T(t^* - \varepsilon) = \Gamma^T(0)$ for any $0 < \varepsilon < t^*$

The trailing candidate can only win the election by choosing a level of negativity above some value t*, as shown in Figure 1. Thus, he would never choose any level of negativity between 0 and t*, as this would give him a lower reputation than if he were entirely positive without increasing his chance of winning the election. The frontrunner will always win the election if her level of negativity is higher than f*, also shown in Figure 1. She would never choose a higher level of negativity than f*, as this would give her a lower reputation than with f* without increasing her chance of winning the election.

Second, as shown in Lemma 3, in any mixed strategy equilibrium, the probability of an electoral tie must be zero.

Lemma 3: $\gamma^T(t)\,\gamma^F(m^F(t)) = 0$, any $t^* \le t \le 1$.

A tie can only occur with positive probability if each candidate puts positive probability on strategies whose combination lies on the $\Delta = 0$ locus. Each candidate would gain by deviating from such a strategy by moving that atom to a slightly more negative strategy, which would discretely increase his probability of winning the election. Thus, this cannot be an equilibrium.

Third, as shown in Lemma 4, if one candidate's cdf has an atom at some strategy i', then the other candidate must put no probability weight in some interval just below the corresponding strategy on the $\Delta = 0$ locus, $m^J(i')$.

Lemma 4:

(a) If $\gamma^F(f') > 0$, any f' with $0 < f' \le f^*$, then there exists some $\delta' > 0$ such that

$\Gamma^T(m^T(f')) = \Gamma^T(m^T(f') - \delta')$

(b) If $\gamma^T(t') > 0$, any t' with $t^* < t' \le 1$, then there exists some $\delta' > 0$ such that

$\Gamma^F(m^F(t')) = \Gamma^F(m^F(t') - \delta')$

A strategy for a candidate that is slightly more positive than one yielding a tie (making Δ=0) causes a small increase in reputation for the candidate, but the probability of winning the election falls discretely from ½ to 0. Thus, for a candidate to be indifferent to a strategy that leads to a tie, he must choose an amount of negativity discretely lower than that yielding the tie, so that the direct increase in reputation is balanced by the decrease in the expected bonus. No probability weight, then, will be placed in the interval between the amount of negativity yielding a tie and the discretely lower amount yielding equal utility for the candidate.

Fourth, as shown in Lemma 5, if one candidate has an interval in which no probability weight is placed, then the other candidate places no probability weight in the corresponding interval on the $\Delta = 0$ locus. In Figure 1, the interval between a and b on the f axis corresponds to the interval between a' and b' on the t axis.

Lemma 5: If there exists an interval (a, b) with $0 < a < b < f^*$, and $\Gamma^F(b - \varepsilon) = \Gamma^F(a)$ for all $0 < \varepsilon < b - a$, then $\Gamma^T(m^T(b - \varepsilon)) = \Gamma^T(m^T(a))$.

If F puts no probability weight in the interval, then T’s probability of winning is the same at any point in T’s corresponding interval. T, then, strictly prefers any strategy with less negativity in the interval because this will increase his reputation without decreasing his expected bonus from winning the election.

Fifth, as shown in Lemma 6, neither candidate has an atom on the interior of the intervals in which they may place probability weight.

Lemma 6: $\gamma^F(f) = 0$ for all f with $0 < f < f^*$ and $\gamma^T(t) = 0$ for all t with $t^* < t < 1$.

If F had an atom at some f between 0 and f*, then from Lemmas 3 and 4, T would not have an atom at $m^T(f)$ and puts no probability weight in some interval below $m^T(f)$. Then F would have a higher expected payoff at $f - \varepsilon$ than at f, since the probability of winning would be the same but the direct effect of being more positive would raise F's reputation. A similar argument holds if T were to have an atom between t* and 1.

Next, as shown in Lemma 7, each candidate has one interval in which they are willing to put some probability weight, and within this interval, there is no range with zero probability weight. That is, each cdf is strictly increasing over the relevant interval and nowhere else.

Lemma 7: There exist f′ and t′, with $0 < f' < f^*$, $t^* < t' < 1$, and $t' = m^T(f')$ such that:

(i) $\Gamma^F(f') = 1$ and $\Gamma^F(b) > \Gamma^F(a)$ for any $0 < a < b < f'$,and

(ii) $\Gamma^T(t') = 1$ and $\Gamma^T(b) > \Gamma^T(a)$ for any $t^* < a < b < t'$.

Assume that F has an interval (a, b) in which there is no probability weight, but F does have weight above b. Then T puts no weight in the corresponding interval from Lemma 4, and neither have atoms near those intervals from Lemma 6. Then T's cdf is continuous near $m^T(b)$, which makes F's expected payoffs continuous near b. Since F's probability of winning is the same at a and b but expected payoffs are strictly greater at a than b, they are also strictly greater at $b + \varepsilon$, for small ε . This contradicts F putting probability weight at any $b + \varepsilon$.

Finally, as shown in Lemma 8, the only possible atoms for each candidate are at completely positive campaigns. While the trailing candidate may or may not have an atom at a completely positive campaign, the frontrunner will always have an atom there.

Lemma 8: The only atom for F is at f = 0 with $\gamma^F(0) \geq [E_{\tilde{f}}[\theta^T(0,\tilde{f})] - E_{\tilde{f}}[\theta^T(t^*,\tilde{f})]] / B_T > 0$. The only possible atom for T is at t = 0.

From Lemmas 4, 6, and 7, the only possible places for atoms are at 0 for F, and 0 and t* for t. T puts probability weight near t* so must receive at least as great an expected payoff near t* as at 0. For this to hold, F must have an atom at 0 since if not, the probability of winning at 0 and t* would be the same, with 0 then having the higher payoff. The atom at 0 for F rules out one at t* for T given Lemma 3.

Our characterization of the cdfs for the two candidates follows from these results, and is given in Theorem 1. As we point out below, however, this is not an explicit solution for them.

Theorem 1: If an equilibrium exists, then there exist f' and t' with $\Delta(f', t') = 0$ such that the equilibrium cdfs satisfy the following:

$$\Gamma^F(f) = 1 + [E_{\tilde{f}}[\theta^T(t',\tilde{f})] - E_{\tilde{f}}[\theta^T(m^T(f),\tilde{f})]]/B_T, \; 0 \leq f \leq f'$$

$$\Gamma^F(f) = 1, \; f' \leq f \leq 1$$

$$\gamma^F(0) = 1 + \left[E_{\tilde{f}}[\theta^T(t',\tilde{f})] - E_{\tilde{f}}[\theta^T(t^*,\tilde{f})]\right]/B_T$$

$$\Gamma^T(t) = \gamma^T(0), \; 0 \leq t \leq t^*$$

$$\Gamma^T(t) = 1 + [E_{\tilde{t}}[\theta^F(f', \tilde{t})] - E_{\tilde{t}}[\theta^F(m^F(t), \tilde{t})]]/B_F, \ t^* \le t \le t'$$

$$\Gamma^T(t) = 1, \ t' \le t \le 1$$

$$\gamma^T(0) = 1 + \left[E_{\tilde{t}}[\theta^F(f', \tilde{t}) - E_{\tilde{t}}\left[\theta^F(0, \tilde{t})\right]\right]/B_F$$

In addition, one of the following must hold:

(a) $E_{\tilde{f}}\left[\theta^T\left(t', \tilde{f}\right)\right] = E_{\tilde{f}}\left[\theta^T\left(0, \tilde{f}\right)\right] - B_T$ and $E_{\tilde{t}}[\theta^F(f', \tilde{t})] \ge E_{\tilde{t}}[\theta^F(0, \tilde{t})] - B_F$, or

(b) $E_{\tilde{t}}[\theta^F(f', \tilde{t})] = E_{\tilde{t}}[\theta^F(0, \tilde{t})] - B_F$ and $E_{\tilde{f}}\left[\theta^T\left(t', \tilde{f}\right)\right] \ge E_{\tilde{f}}\left[\theta^T\left(0, \tilde{f}\right)\right] - B_T$

The cdf for the frontrunner has an atom at 0, then is strictly increasing up to f', where it equals 1. On the strictly increasing region, the rate of increase of Γ^F equals $-E_{\tilde{f}}\left[\frac{\partial\theta^T}{\partial t}\frac{\partial M^T}{\partial f}\right]/B_T$. Thus, the additional weight the frontrunner puts on more negative strategies equals the magnitude of the expected rate at which the trailing candidate's expected reputation falls with an increase in the trailing candidate's own negativity. This increase ends at f' when Γ^F equals 1. The remaining probability weight is then placed in the atom at 0. Similarly, in the strictly increasing region of the trailing candidate's cdf, the rate of increase of Γ^T equals $-E_{\tilde{t}}\left[\frac{\partial\theta^F}{\partial f}\frac{\partial M^F}{\partial t}\right]/B_F$. This region begins at t* and ends at t' when Γ^T equals 1. If there is any remaining probability weight, it is placed in an atom at 0.

As noted above, the trailing candidate may or may not have an atom at 0. In the interior of the parameters consistent with Condition (a), the trailing candidate does have an atom there. For this to be true, the trailing candidate must be indifferent between a completely positive campaign and one with the maximum negativity in the relevant interval, t'. On the interior of the parameters satisfying Condition (b), the trailing candidate receives a higher payoff at t' than at 0 so cannot have an atom at 0. On the boundary between Conditions (a) and (b), the trailing candidate is indifferent between running a completely positive campaign and one at t', but does not have an atom at 0. To get T's indifference in (a), $\gamma^F(0)$ must be at the lower bound given in Lemma 8. In (b), however, $\gamma^F(0)$ can be above the lower bound, since this lowers T's payoff at 0 negativity.

Although the $\Gamma^I(i)$ functions given in Theorem 1 characterize the equilibrium mixed strategies, they are not explicit solutions. $\Gamma^F(f)$ depends on an expectation taken with respect to probabilities specified by the derivative of $\Gamma^F(f)$. To actually find the equilibrium cdfs, a fixed point is needed. Assuming a $\Gamma^F(f)$ and substituting it into the expectations in Theorem 1 yields a new $\hat{\Gamma}^F(f)$. To be an equilibrium, this $\hat{\Gamma}^F(f)$ must be the same as the function initially posited. Such a fixed point may or may not exist, depending on whether an (f',t') pair exists under which one of Condition (a) or (b) is satisfied. When neither condition holds so that a fixed point does not exist, the game has no equilibrium. However, a discrete approximation to the continuous game would have an equilibrium.

Since an equilibrium does not always exist in the continuous game, it is interesting to see how the sufficient conditions in Theorem 5 of Dasgupta and Maskin (1986) are violated. Several of their conditions are always met. In particular, the strategy spaces are the [0, 1] closed interval, the utility functions are bounded, and the subset of discontinuities (the $\Delta = 0$ curve) is well behaved. Their conditions also require that $U^F + U^T$ is upper semi-continuous, which is satisfied when $B_T = B_F$. When $B_T \neq B_F$, $U^F + U^T$ is not upper semi-continuous, but this violation may not be crucial, since it is not clear that any inequality between B_T and B_F will cause both (a) and (b) in Theorem 1 to fail to be satisfied. Their condition that payoffs must be weakly lower semi-continuous is more significant for our results. While U^F is always weakly lower semi-continuous in f at all relevant t, U^T is weakly lower semi-continuous everywhere except at the endpoint of the $\Delta = 0$ curve where t = 1 and f = f*. At that point, U^T is not left lower semi-continuous in t, since its value at t = 1 is greater than the limit of U^T as t approaches 1 from below. This follows since T wins B half the time at (f*, 1) but always loses the election when t is less than 1.[4] Interestingly, as shown in the example below, this single violation of their conditions is sometimes enough to lead to nonexistence, even when $B_T = B_F$ and all other of their conditions are satisfied.

While it is difficult in general to determine when there is no equilibrium or to specify it when there is, for some specific θ^I functions, it is possible to explicitly solve for the equilibrium cdfs, as in the example given in the next section. In this example, we find the solution when it exists. When it does not exist, we solve numerically for the equilibrium in a discrete approximation to the continuous game.

5. A solvable example

Consider an example where the marginal effects of advertising on reputation are constant, where the θ functions differ only because the candidates may have different resources to spend on their campaigns, and the values of B are the same for the two candidates. Let G be the direct gain to a candidate's reputation from running a single

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[4] To check if $U^T(t, f)$ is weakly lower semi-continuous at any (f, $m^T(f)$), check the limits of $U^T(t, f)$ as t approaches $m^T(f)$ from above and below. If $U^T(m^T(f), f)$ is greater than both of these limits, then $U^T(t, f)$ is not weakly lower semi-continuous there. The point (f*, 1), however, is an endpoint of the $\Delta = 0$ curve and only the limit from below is defined. To be weakly lower semi-continuous there, that limit must be at least as great as $U^T(1, f^*)$.

positive ad, and let L be the loss in reputation due to an opponent's single negative ad. The candidates are equally effective in their ads so G and L are the same for both candidates. The candidates have different resources to spend on their campaigns, denoted as R_F and R_T. The overall effect is the number of ads of each type multiplied by the effectiveness of an ad. That is, the effect on candidate I's reputation from positive campaigning is $GR_I(1 – i)$, and the effect on the opponent's reputation from I's negative campaigning is LR_Ii. In addition, there is a "meltdown" effect that occurs when both candidates choose negative campaigns; this reduces the candidates' reputations by some amount D multiplied by the product of the amounts of negative campaigning they do, R_FfR_Tt.[5] The type of meltdown depends on the sign of D: when D is positive, voters are angered by negative campaign wars, resulting in much lower reputations for both candidates. When D is negative, negative advertising wars cause voters to become apathetic, so that negative ads are largely ignored and thus have smaller effects on reputation. This yields the following functional form for the effect of campaigning on reputation:

$$\theta^I(i, j) = GR_I(1 - i) - LR_Jj - DR_IiR_Jj \qquad (6)$$

[5] This differs from the "boomerang" effect of Skaperdas and Grofman (1995), which is the amount by which a negative ad hurts the advertiser, regardless of the action of the opponent. Our "meltdown" is voter dissatisfaction with a negative campaign war, in which both candidates go negative.

Then $\theta^I(0, 0) = GR_I$, $\theta^I(1, 1) = -LR_J - DR_IR_J$, $\theta^I(1, 0) = 0$, and $\theta^I(0, 1) = GR_I - LR_J$.[6] For this function, the conditions (1), (2), (4), and (5) under which there is no pure strategy equilibrium are straightforward as given in the following conditions:

$$G\max\{R_F, R_T\} < L\min\{R_F, R_T\} \qquad (7)$$

$$\max\{0, -DR_FR_T\} < G\min\{R_F, R_T\} \qquad (8)$$

$$GR_T < B \qquad (9)$$

$$\max\{0, L(R_T - R_F)\} < X_F - X_T < LR_T - GR_F \qquad (10)$$

(7) and (8) ensure that conditions (1) and (2) are satisfied, (9) implies that (4) is satisfied, and (10) ensures that the assumptions that $\Delta(0, 0) > 0$, $\Delta(0, 1) < 0$, and $\Delta(1, 1) > 0$ hold. For these θ^I, $\Delta(f, t) = X_F - X_T + G(R_F(1 - f) - R_T(1 - t)) + L(R_Ff - R_Tt)$. Setting $\Delta = 0$ and solving yields $t = \left(\frac{R_F}{R_T}\right) f + \frac{X_F - X_T + G(R_F - R_T)}{(L-G)R_T} = m^T(f)$. Then

$$t^* = 1 - \left(\frac{R_F}{R_T}\right) f^* = \frac{X_F - X_T + G(R_F - R_T)}{(L-G)R_T} \qquad (11)$$

The inequalities in (10) ensure that t* and f* are between 0 and 1. Note that (7) to (10) and t* are homogeneous of degree 0 in G, L, D, B, and $X_F - X_T$. Hence, without loss of

[6] This $\theta^I(i, j)$ function is consistent with the assumptions in the discrete decision model in Fletcher and Slutsky (2010) with the addition of differential resources for the candidates.

generality, G can be set equal to 1 with the magnitudes of all the reputation variables taken relative to the effect on reputation of a single positive ad. Condition (7) is then equivalent to assuming that $L > 1$, $R_T > 0$, and $R_T/L < R_F < LR_T$.

Condition (7) implies that negative campaigning increases the chance of winning the election, because negative campaigning by the poorer candidate lowers the opponent's reputation more than positive campaigning by the richer candidate raises that candidate's own reputation. Condition (8) implies that even if $D < 0$, so that meltdown takes the form of voter apathy, this apathy is not too extreme.[7] Condition (9) means that winning the election is important to the trailing candidate: the bonus from winning is larger than the boost to reputation from running an entirely positive campaign, so that T is not a pure gentleman politician. Finally, condition (10) imposes that the election is contestable between the candidates, since the difference in their initial reputations is small enough that campaign activities can change the outcome.

Lemma 9 presents expressions for the means of the distributions of f and t, $\mu_i \equiv E_{\tilde{\imath}}[\tilde{\imath}]$, that are consistent with the properties for the cdfs as given in Theorem 1.

Lemma 9: In an equilibrium, the following must hold:

$$\mu_F = \frac{G\mathrm{R_F}(f')^2}{2B - D(\mathrm{R_F})^2(f')^2}$$

[7] The idea behind negative D is that apathy reduces the effectiveness of ads because voters stop paying attention. The effect of ads should be damped but not changed in direction. If D were below this bound, a candidate's own reputation would actually increase from running more negative ads about the opponent.

$$\mu_T = \frac{G R_T((t')^2 - (t^*)^2)}{2B - D(R_T)^2\,((t')^2 - (t^*)^2)}$$

Since (f′, t′) is a point on the Δ=0 curve, they are related with $(\frac{R_F}{R_T})f' = t' - t^*$.

Therefore, given all parameters, both means and, hence, both cdfs are determined by the endogenous variable t′. Thus, there exists a family of pairs of cdfs that depend on only one variable. A Nash equilibrium exists if in this family there exists a t' between t* and 1 which also satisfies (a) or (b) of Theorem 1. Theorem 2 specifies the parameter values at which such a t' exists, gives its unique value in those cases, and gives the parameter values under which no such t' exists.

Theorem 2: Assume that campaigns affect reputation according to the function in (6) with $0 < G < L$, $0 < GR_T < B$, $GR_T/L < R_F < LR_T/G$, $-D < \min\{G/R_F, G/R_T\}$, and $\max[0, L(R_T - R_F)] < X_F - X_T < LR_T - GR_F$.

(I) If a Nash equilibrium exists in mixed strategies, the cdfs for the two candidates must be the following:

$\gamma^F(0) = 1 - \alpha f'$

$\Gamma^F(f) = 1 + \alpha(f - f'),\ 0 \le f \le f'$

$\Gamma^F(f) = 1,\ f' \le f \le 1$

$\Gamma^T(t) = \gamma^T(0) = 1 - \beta(t' - t^*),\ 0 \le t \le t^*$

$\Gamma^T(t) = 1 - \beta(t' - t),\ t^* \le t \le t'$

$\Gamma^T(t) = 1,\quad t' \le t \le 1$

where $\alpha \equiv \frac{2GR_F}{2B - D(R_F)^2 (f')^2}$ and $\beta \equiv \frac{2GR_T}{2B - D(R_T)^2\,((t')^2 - (t^*)^2)}$

This Nash equilibrium exists in either of the following circumstances:

(i) $B \leq \frac{GR_T}{2}(3 - t^*)$ and $\frac{2(B-GR_T)}{(R_T)^2(1-t^*)^2} \leq D \leq \frac{3G^2}{2(B-GR_T t^*)}$

with t′ = $(\frac{R_F}{R_T})f'$+ t* = t* - $\frac{G}{DR_T} + \sqrt{\left(\frac{G}{DR_T}\right)^2 + \frac{2(B-GR_T t^*)}{DR_T{}^2}}$

(ii) $max\left[\frac{3G^2}{2(B-GR_T t^*)}, \frac{2(B-GR_T(1-t^*))}{R_T{}^2(1-(t^*)^2)}\right] \leq D,$

with t′ = $(\frac{R_F}{R_T})f' + t^* = -\frac{G}{DR_T} + \sqrt{(\frac{G}{DR_T} + t^*)^2 + \frac{2B}{DR_T{}^2}}$

(II) There is no Nash equilibrium in the following circumstance:

(iii) $D < min\left[\frac{2(B-GR_T(1-t^*))}{R_T{}^2(1-(t^*)^2)}, \frac{2(B-GR_T)}{R_T{}^2(1-t^*)^2}\right]$

Γ^F and Γ^T are shown in Figure 2 for case (i), which corresponds to (a) in Theorem 1 where T has an atom at 0. Case (ii) corresponds to Theorem 1(b) in which T has no atom. In either case, since the cdf of Γ^F is linear on the range (0, f'], the pdf is uniform on that range. Similarly, the cdf of Γ^T is linear on the range [t*, t'].

The description of these existence and nonexistence regions is straightforward in terms of D with the values of the other parameters held fixed. Nonexistence occurs when D is small. Either there is apathy or a relatively small amount of anger at negative campaign wars. Existence region (i), where T has an atom at zero negativity, occurs for a moderate range of anger, while existence region (ii), where T has no atom, occurs for more extreme anger. This indicates that greater anger at negative campaigning does not necessarily drive the candidates, especially T, to be positive.

Alternatively, consider how the nature of equilibria varies with B for fixed other parameters. For small and moderate B there exist equilibria in regions (i) and (ii), respectively. However, at large values of B no equilibrium exists. B large means that candidates are approaching being Lombardians, caring much more about winning than about other reputational effects. Since a standard assumption in the literature is that the goal of candidates is to maximize their probability of winning, it is significant that nonexistence would occur for such candidates, and it is important to consider how they would behave when no equilibrium exists in the continuous game. This is analyzed in the next section.

6. Equilibria in discrete approximations of the game

For the example in Section 5, both existence and nonexistence of a Nash equilibrium are possible, and the equilibrium can be found analytically when it exists. To gain insight into how the candidates might behave when there is no equilibrium, we solve discrete approximations of the game. Actual candidates do solve a discrete game, as even when they buy large numbers of ads, they are choosing how many of them will be negative. In some sense, then, the continuous game is really the approximation and the discrete game is reality.

That said, the finest grid for which we can consistently solve is far coarser than most campaigns in reality. We use the existence case as a benchmark to confirm that our finest grid yields results that are close to those in the continuous game. We then use that grid fineness to examine how candidates behave in equilibrium in the discrete approximations in the nonexistence region.

Gambit (McKelvey, McLennan, Turocy (2014)) is a software program that numerically solves games of different forms, including finite games in the strategic form.

We use Gambit to find the solutions for several examples of our discrete games. Table 1 compares important aspects of the equilibria of the continuous and discrete games, and Figure 3 shows an existence example from region (i), with a grid size of 101x101 (i.e. the players can choose the fraction negative in increments of 0.01).[8] As the Table and Figure show, the pdf of the discrete approximation in the 101x101 game is strikingly similar to the one for the continuous game. $\gamma^F(0)$, the height of the frontrunner's atom at zero, is 0.53 in the continuous game, while it is 0.54 in the discrete game. $\gamma^T(0)$, the height of the trailing candidate's atom at zero, is 0.13 in the continuous game and 0.12 in the discrete game.

The supports are also very similar. In the continuous game equilibrium, the frontrunner puts probability weight from 0 to f', which equals 0.3483 in this game, while the trailing candidate puts weight between t*, which equals 0.50, and t′, which equals 0.9354. In the equilibrium of the discrete approximation game, F puts weight at 0 at the bottom of the distribution and at 0.35 at the top. In the discrete game, for T, the lower end of the support, which we define as $\bar{t}$, is at 0.51 and the upper end t′ is at 0.93.

This example is representative of what seems to occur. The supports in the discrete approximation games are the grid points next to the values of those bounds in the continuous game. For T, the value of $\bar{t}$ always seems to be the grid point above t*, and t′ is the grid point below that in the continuous game. For F, f' can be the grid point above or below the corresponding value in the continuous game. Recall that in the continuous game, the probability distributions for both players are uniform over the continuous parts of their distributions. As shown in Figures 3 and 4, the discrete game equilibria

[8] This example has G = 1, L = 2, B = 4.5, $X_F - X_T = 1$, $R_F = 5$, $R_T = 4$, and D = 0.5. 101x101 is the finest grid for which the current version of Gambit can consistently find the set of all Nash equilibria in the game.

approximate this. In the discrete game, the grid points often alternate between having probability weight and no probability weight. For those that do have probability weight, the probabilities vary some but are approximately constant.

Figure 4 shows the pdfs for an example of a 101x101 game with parameters satisfying region (ii).[9] As before, this pdf is very similar to the continuous one, and as expected in this region, the trailing candidate has no atom at zero negativity.

Given how consistent these approximations are for the existence cases, it seems reasonable to use them to explore candidate behavior in the nonexistence region. Two such pdfs are shown in Figure 5, and important aspects of the equilibria are given in Table 2.[10] In both examples, the trailing candidate's pdf looks very much like the one from the existence regions: there may or may not be an atom at zero, and there is a roughly uniform region between t* and t′. The single important difference is that unlike what we see in the existence regions, t′ = 1, so that the trailing candidate puts probability weight all the way up to complete negativity in every nonexistence example we have constructed.

The frontrunner's strategy is also not quite the same as in the existence regions. As before, the frontrunner's pdf has an atom at zero, then a roughly uniform distribution up to a point analogous to f′ (which is now approximately f*) in the previous Figures. Now, however, F has a second atom at the gridpoint immediately above f*.[11] This is consistent with the point of discontinuity where the conditions of Dasgupta and Maskin

[9] This example has G = 1, L = 1.5, B = 15, $X_F - X_T = 2$, $R_F = 15$, $R_T = 12$, and D = 1.
[10] The first nonexistence example has D < 0, with G = 1, L = 1.1, B = 20, $X_F - X_T = 0.1$, $R_F = R_T = 10$, and D = -0.05. The second nonexistence example has G = 1, L = 1.5, B = 75, $X_F - X_T = 2$, $R_F = 15$, $R_T = 12$, and D = 3.
[11] In both of the nonexistence examples, T's atom at zero negativity is larger than F's, but this need not always be true, and depends on the parameters chosen.

(1986) are violated, as discussed in section 4. Putting probability weight above f* is ruled out in the continuous case because it is a dominated strategy: the frontrunner can always do better by choosing a strategy closer to f*. In the discrete case, however, F cannot get closer to f* than the gridpoint immediately above it, so there is no strategy that dominates putting weight on that gridpoint. At that additional atom, F is sufficiently negative to always win the election, regardless of the negativity of T's campaign.

Both candidates, then, tend to be more negative in the discrete approximations when the continuous game has no equilibrium than they are in the games in which a continuous equilibrium exists. For T, this occurs because he puts probability weight all the way up to running a completely negative campaign, instead of only going up to a t′ that is less than 1. It occurs for F because the atom at 0 negativity often seems smaller than when a continuous equilibrium exists, and because of the second atom just above f* at the top of F's distribution.

7. Properties of equilibria

7.1 Differences between F and T

One important question about negative campaigning is whether the trailing candidate or the frontrunner is typically more negative. A standard view is that the trailing candidate goes negative first and more extensively. In our model, the trailing candidate is often more negative, but that is not always the case.

Theorem 3: Under the parameter values at which a mixed strategy equilibrium exists, $\gamma^T(0) < \gamma^F(0)$ and $\mu_F < \mu_T$. The trailing candidate has a smaller atom of being purely positive and the frontrunner runs a less negative campaign on average.

This result corresponds to those in Skaperdas and Grofman (1995) and Harrington and Hess (1996). However, the equilibria in the discrete approximation game when no equilibrium exists in the continuous game can have very different properties than those given in Theorem 3. As shown in the examples in Table 2, $\gamma^T(0)$ can now be greater than $\gamma^F(0)$. This combined with the existence of an atom for F just above f* means that it is possible that μ_F can exceed μ_T, and that the magnitude of the difference can be large. The first example in Table 2 illustrates this, where F is more than three times as negative on average as is T. Recall that the non-existence region for the continuous game includes important and realistic cases where B is large, so that winning is the predominant motive of candidates. In that circumstance, frontrunners could run more negative campaigns than their opponents. It should be noted that in nonexistence example 1, $R_F = R_T$. The counterintuitive result that $\mu_T < \mu_F$ is not due to T having additional resources to become the frontrunner.

A second important question is: what are the probabilities that each candidate wins the election? As shown in the next result, the apparently disadvantaged candidate can have the higher probability of winning. Let π_T and π_F denote the probabilities with which T and F win the election.

Theorem 4: The difference in the probabilities of T and F winning the election is:

$$\pi_T - \pi_F = \gamma^F(0)(1- \gamma^T(0)) - \gamma^T(0)$$

This is positive in continuous existence region (ii) when $\gamma^T(0) = 0$. In region (i) when $\gamma^T(0) > 0$, this can be positive or negative.

Which candidate has the greater probability of winning depends only on the magnitudes of the atoms at 0 negativity. This follows since, as shown in the proof, the continuous parts of the distributions are chosen in a way that gives the candidates exactly equal average probabilities of winning when both play from them. When a candidate runs an entirely positive campaign, she has a lower probability of winning but is willing to do this because it leads to a direct boost in reputation. The probability of winning is endogenous to the equilibrium strategy choices and does not necessarily correspond to which player has more initial advantages. In line with this, note that $\pi_T > \pi_F$ can occur even with $R_F = R_T$. The trailing candidate has the greater probability of winning due to running a relatively negative campaign, and not from having more resources to overcome the initial reputation disadvantage.

Consider the equilibrium of the discrete approximation game when an equilibrium does not exist in the continuous game. Theory suggests that pdfs of F and T will be uniform over the (0, f*) and (t*, 1) intervals respectively, just as in the continuous game equilibria, so that the opponent will be indifferent over gridpoints in the relevant range. The numerical solutions yield essentially this.[12] Thus, over these intervals the two players will have an approximately equal chance of winning. The difference in their winning probabilities will again depend on the magnitude of their different atoms. The

[12] In those intervals, when a gridpoint receives weight, the weights are relatively constant. However, roughly every other grid point receives no weight.

only qualitative difference is that F now has an atom at the grid point immediately above f*, which we denote as $\gamma^F(f^* + \varepsilon)$. This increases the probability that F wins the election since F always wins when playing this atom. The value of π_T - π_F is now $(\gamma^F(0) - \gamma^F(f^* + \varepsilon))(1- \gamma^T(0)) - \gamma^T(0)$, which can be negative even when $\gamma^T(0)$ is at or near 0, if $\gamma^F(f^* + \varepsilon)$ is greater than $\gamma^F(0)$. In both examples given in Table 2, π_F exceeds π_T.

7.2 Comparative statics

Next, consider some comparative statics for the equilibrium strategies when an equilibrium exists in the continuous game. Table 3 shows the signs of changes in the average negativity of F and T in the two existence regions (i) and (ii). There exist a number of apparently counterintuitive results. Generally, one would expect that if a candidate were stronger either by having a greater initial reputation or by having more resources, then that candidate would be less negative and the opponent would be less positive. This is true for F in any equilibrium and for T in the equilibrium in region (ii) where $\gamma^T(0) = 0$. However, in the region (i) equilibria where $\gamma^T(0) > 0$, there are circumstances where T being stronger causes μ_T to increase and μ_F to decrease. It should be noted that all of the comparative statics results for μ_T are ambiguous in this region, so this result presumably arises because of effects of the parameter changes on $\gamma^T(0)$.

An increase in B would make winning more important relative to enhancing reputation. This would tend to increase the incentives to be negative, since that could raise the probability of winning. Except for the ambiguity for μ_T in region (i), increasing B raises the average negativity of the players.

It seems reasonable to conjecture that an increase in G would make players less negative in the equilibrium. Surprisingly, this turns out not to be generally true. The effect of G on both μ_T and μ_F in both regions is ambiguous. An increase in G can lead players to be more – not less – negative. These ambiguities may not be as counterintuitive as might first appear, however. Holding fixed the actions of the opponent, an increase in G in a very stylized sense is analogous to a drop in the price of being positive. This will have an income effect as well as a substitution effect. Think of the player as consuming two goods: positive reputation gains ($P = GR_F(1 – f)$ for F) and negative reputation effects on the opponent ($N = LR_Ff$). There is a linear budget line between P and N whose slope is -L/G. A rise in G rotates this budget line out, just as would a price drop in a standard consumer optimization problem. The substitution effect would imply an increase in P and a drop in N, which would imply a drop in f. The candidate is also wealthier, which would imply a desire to increase both P and N, if both are normal goods. If the cross effect on N leads to a desired net increase, then this can only occur by increasing μ_F. There are additional effects since G alters the behavior of the opponent, which enters into the expected utility function of the candidate in various ways and also has ambiguous effects.

It would also seem that increasing L would cause the candidates to be more negative, since negative ads become more effective. This turns out to be true for F but not generally for T. Not only is there an ambiguity for T in region (i), but in region (ii) the effect is unambiguously the reverse: increasing L lowers the average negativity of T's campaign. This may again be explained by an implicit income effect. In that region, T has the greater probability of winning, so may choose to use some of the implicit extra resources created by the increase in L to enhance his own reputation by increasing P without lowering N.

The expected effect for changes in D also does not always occur. In the continuous existence cases D is positive, so voters are angered by negative campaign wars, reducing the reputations of both candidates. Since this term affects the candidates symmetrically it does not alter the probability of winning. Since candidates value their reputations beyond winning, it would seem reasonable that increasing D would reduce the average negativity of their campaigns. This happens unambiguously only for F in region (i). In all the other cases, it is at least possible that raising D makes the candidates run more negative campaigns.

One final interesting comparative static result is the effect of an increase in a candidate's initial reputation on the expected payoff of that candidate. This is the focus of the analysis of Fletcher and Slutsky (2010), who show that having a higher initial reputation, X_F or X_T, could lower the candidate's post-election payoff. One interpretation of this is that an incumbent may gain by governing poorly, and thereby starting the re-election campaign with a lower initial reputation. In that model, the counterintuitive results are due to global, not local, changes. That is, their basic 2×2 game has different types of equilibria at different parameter values. Within an equilibrium type, increasing initial reputation always raises the expected payoff. However, an increase in reputation could shift the equilibrium type and lead to a discontinuous drop in the post-election expected payoff. In some circumstances, a similar result holds in the model here.

Theorem 5: In continuous existence region (ii), increases in X_F or R_F can, and decreases in X_T or R_T do, lower EU^F. In existence region (i), increases in X_T or R_T can, and decreases in X_F or R_F do, lower EU^T.

Note that in this case, the result is a local result and does not arise because the increase induces a change in the nature of the equilibrium. It arises in part because the higher reputation for the particular candidate in each case causes the other candidate to be more negative. The impact of the opponent's extra negativity outweighs the direct reputation gain.

8. Conclusions

In this paper, we develop a model in which political candidates care not only about winning the election but also about their reputations. In contrast to most models of reputation, we focus on how choices in the campaign, and not past behavior, affect reputation. Candidates must choose how negative their campaigns should be, when being more negative can raise the probability of winning but can lower the candidate's reputation directly.

Although the model is relatively simple to formulate, equilibrium behavior becomes quite complex. First, there are almost never pure strategy equilibria. The only pure strategy equilibrium that exists under a range of parameter values is for the candidates to run completely positive campaigns. This occurs only when winning has a relatively insignificant effect on reputation. Negative campaigning will only occur in a mixed strategy equilibrium. This is in contrast to the negative campaigning models of Skaperdas and Grofman (1995) and Harrington and Hess (1996), where negative campaigning occurs in pure strategy equilibria. Having a mixed strategy equilibrium in political contests may be more reflective of reality. Real candidates try to conceal their own strategies from their opponents while paying to learn about their opponents' actions, and even engaging in illegal espionage to get that information. Such behavior would be unnecessary if there were pure strategy equilibria where all such information could be inferred and would not need to be learned.

Second, equilibria do not always exist in pure or mixed strategies in our model, in which campaign negativity is a continuous variable. The situations of nonexistence are not limited or extreme, but include very realistic situations. Two factors which affect whether an equilibrium exists are the extent to which there is "meltdown" from negative campaign wars and how important is winning to candidate reputations. No equilibrium exists if voters are relatively indifferent to negative campaign wars between the candidates. Both being negative has roughly the same effect on reputations as the effects of their negative campaigns separately. We also find that no equilibrium exists if candidates put very high weight on winning relative to the direct reputation effects of the campaign. This approximates the standard assumption that candidates maximize the expected probability of winning. When a continuous equilibrium does exist, we analytically find it in a particular example. When the continuous game has no equilibrium in pure or mixed strategies, we solve finite approximations of the game, which may be a more realistic model of actual candidate strategy spaces.

When the equilibrium does exist, we find that the trailing candidate is, on average, more negative than the frontrunner. This is consistent with results in Skaperdas-Grofman and Harrington-Hess. Because of the increased negativity of the trailing candidate's campaign, that candidate can have a higher probability of winning the election than the frontrunner.

Some of the comparative statics results in this model differ significantly from those of Skaperdas- Grofman. For example, they find that as negative campaigning becomes more effective, both candidates will undertake more negative campaigns. We find a similar result for the frontrunner, but find that, surprisingly, this can cause the trailing candidate to run a more positive campaign. We also find that several parameters that would intuitively seem to have an obvious effect on negativity, such as the effectiveness of positive ads and initial reputation or resource levels, have ambiguous effects for at least one player. One interesting comparative statics result is that a candidate can get a higher expected payoff when starting from a weaker position, either because the candidate's initial reputation or resources are reduced or because the opponent's initial reputation or resources are increased. This is consistent with the result of Fletcher and Slutsky (2010) that bad government can be good politics.

The results for the finite approximation games when there is no continuous game equilibrium can differ significantly. The major qualitative change is that the frontrunner now has an atom at complete negativity as well as at zero negativity. In contrast to the result when a continuous equilibrium exists, the frontrunner can now be more negative on average than the trailing candidate. Since this case includes the realistic cases where winning is overwhelmingly important to candidates, frontrunners being more negative may occur in some actual campaigns, contrary to the results in Skaperdas-Grofman and Herrington-Hess. In turn, this means that in these situations, the frontrunner is more likely to maintain the initial advantage and have the higher probability of winning.

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Table 1: Comparisons of important aspects of continuous and discrete game equilibria, existence regions (i) and (ii)

	(i)		(ii)	
	Continuous	Discrete	Continuous	Discrete
$\gamma^F(0)$	0.5345	0.5452	0.9091	0.9153
$\gamma^T(0)$	0.1292	0.1179	None	None
μ_F	0.0811	0.0810	0.0039	0.0039
μ_T	0.6250	0.6341	0.8870	0.8865
f'	0.3483	0.35	0.0859	0.08
t^*	0.5	0.5	0.8333	0.8333
$\bar{t}$	0.5	0.51	0.8333	0.84
t'	0.9354	0.93	0.9407	0.94

Note: Discrete approximations are made with a 101x101 payoff matrix. $\gamma^F(0)$ and $\gamma^T(0)$ are the height of each player's atom at zero negativity, while μ_F and μ_T are average negativity for each player. t^* is defined in the text. f' and t' are the upper bounds of the supports for the pdfs of F and T, respectively. $\bar{t}$ is the lower bound of T's support.

Table 2: Important aspects of the discrete approximation equilibria for nonexistence examples 1 and 2

	Example 1	Example 2
$\gamma^F(0)$	0.0343	0.5090
$\gamma^T(0)$	0.5960	0.6145
μ_F	0.7525	0.0619
μ_T	0.2242	0.3546
f*	0.9	0.1333
f′	0.9	0.13
f′ + ε	0.91	0.14
t*	0.1	0.8333
$\bar{t}$	0.11	0.84
t′	1	1
$\gamma^F(f' + \varepsilon)$	0.6850	0.3940

Note: Discrete approximations are made with a 101 x 101 payoff matrix. $\gamma^F(0)$ and $\gamma^T(0)$ are the height of each player's atom at zero negativity, while μ_F and μ_T are average negativity for each player. f* and t* are defined in the text. f′ and t′ are the upper bounds of the supports for the pdfs of F and T, respectively, not including any upper atom. f′ + ε is the location of the second atom for F. $\bar{t}$ is the lower bound of T's support. $\gamma^F(f' + \varepsilon)$ is the height of F's atom at the upper bound of his support.

Table 3: Effects of parameter increases on average campaign negativity

	Region (i)		Region (ii)	
	μ_F	μ_T	μ_F	μ_T
X_F	-	+/-	-	+
X_T	+	+/-	+	-
B	+	+/-	+	+
L	+	+/-	+	-
G	+/-	+/-	+/-	+/-
D	-	+/-	+/-	+/-
R_F	-	+/-	-	+
R_T	+	+/-	+	-

Note: μ_F and μ_T are the average negativity of the frontrunner and trailing candidate, respectively. Region (i) is where the trailing candidate has an atom at 0. X_F and X_T are the pre-campaign reputations of the frontrunner and trailing candidate, respectively. B is the value of winning the election. L is the amount by which a negative advertisement lowers the opponent's reputation, and G is the amount by which a positive ad increases the advertising candidate's reputation. D is the "meltdown" effect that occurs when both candidates run negative ads. R_F and R_T are the resources available to the frontrunner and trailing candidate, respectively. +/- means that the effect is positive at some parameter values and negative at others.

Figure 1
Winner of the election as a function of the candidates' mixed strategies

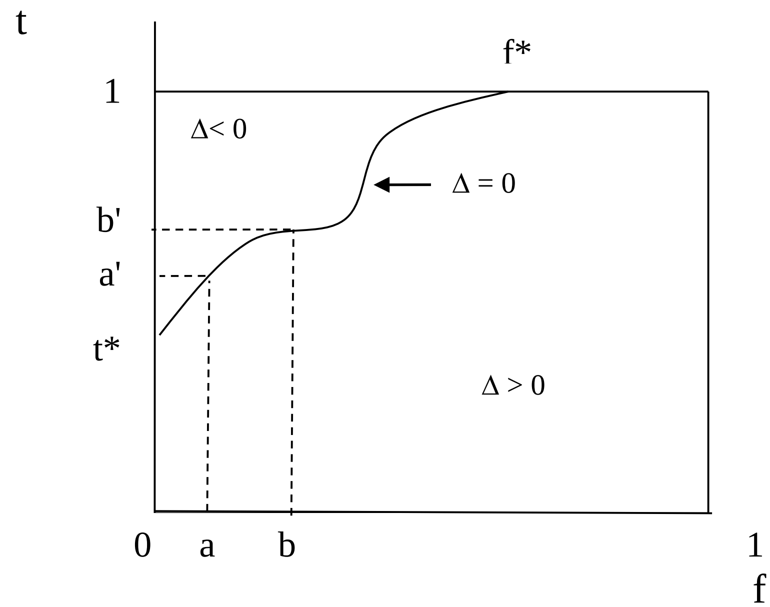


Figure 2
Cumulative distribution functions for campaign negativity

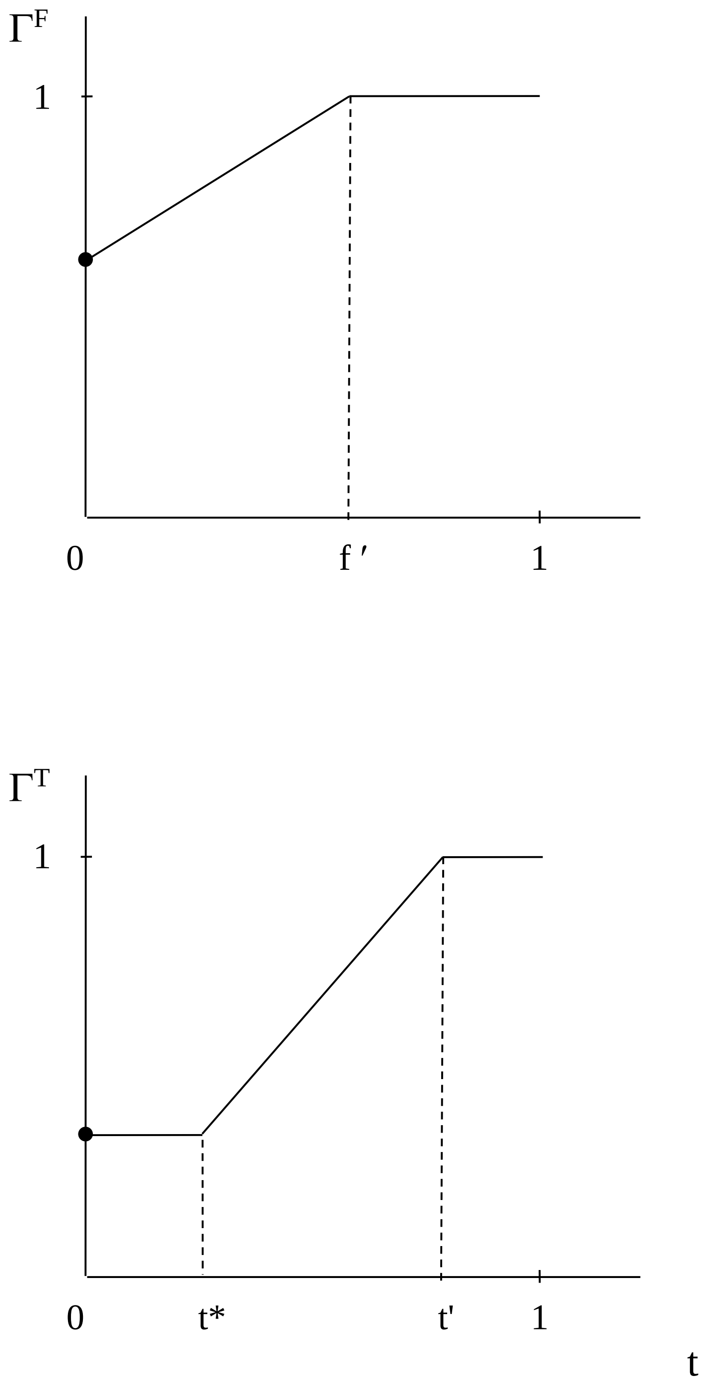
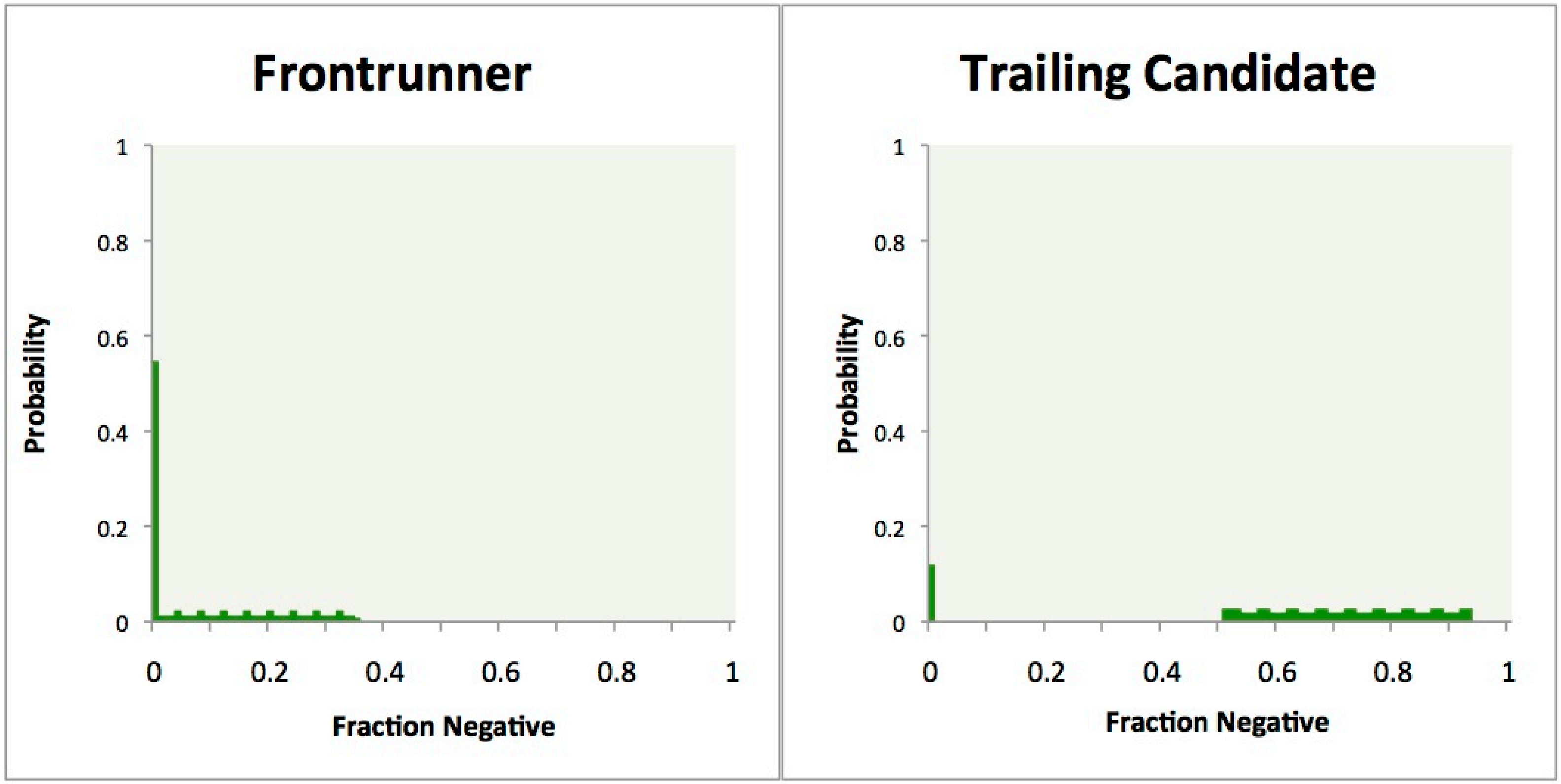
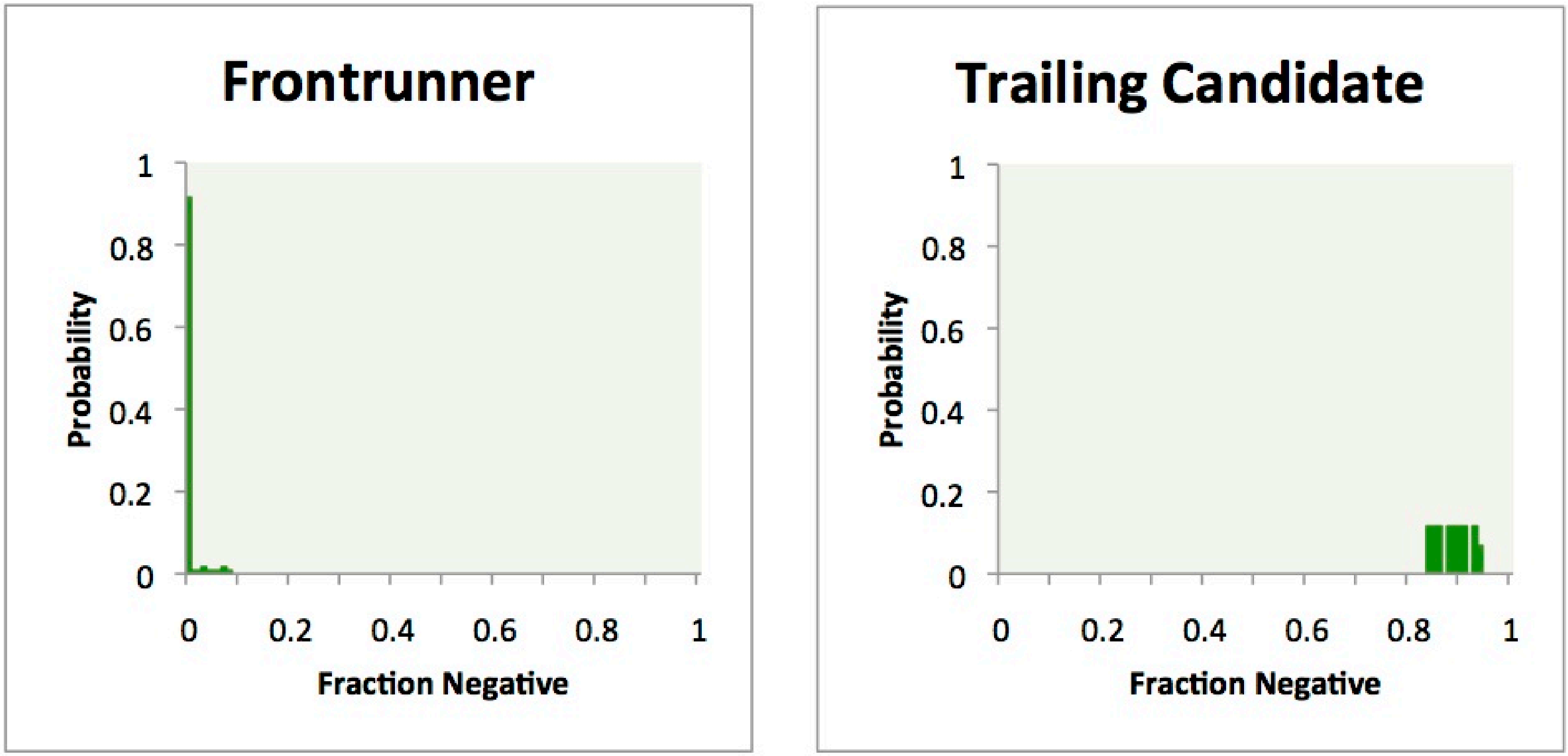


Figure 3: Discrete equilibrium probability distributions, existence region (i)



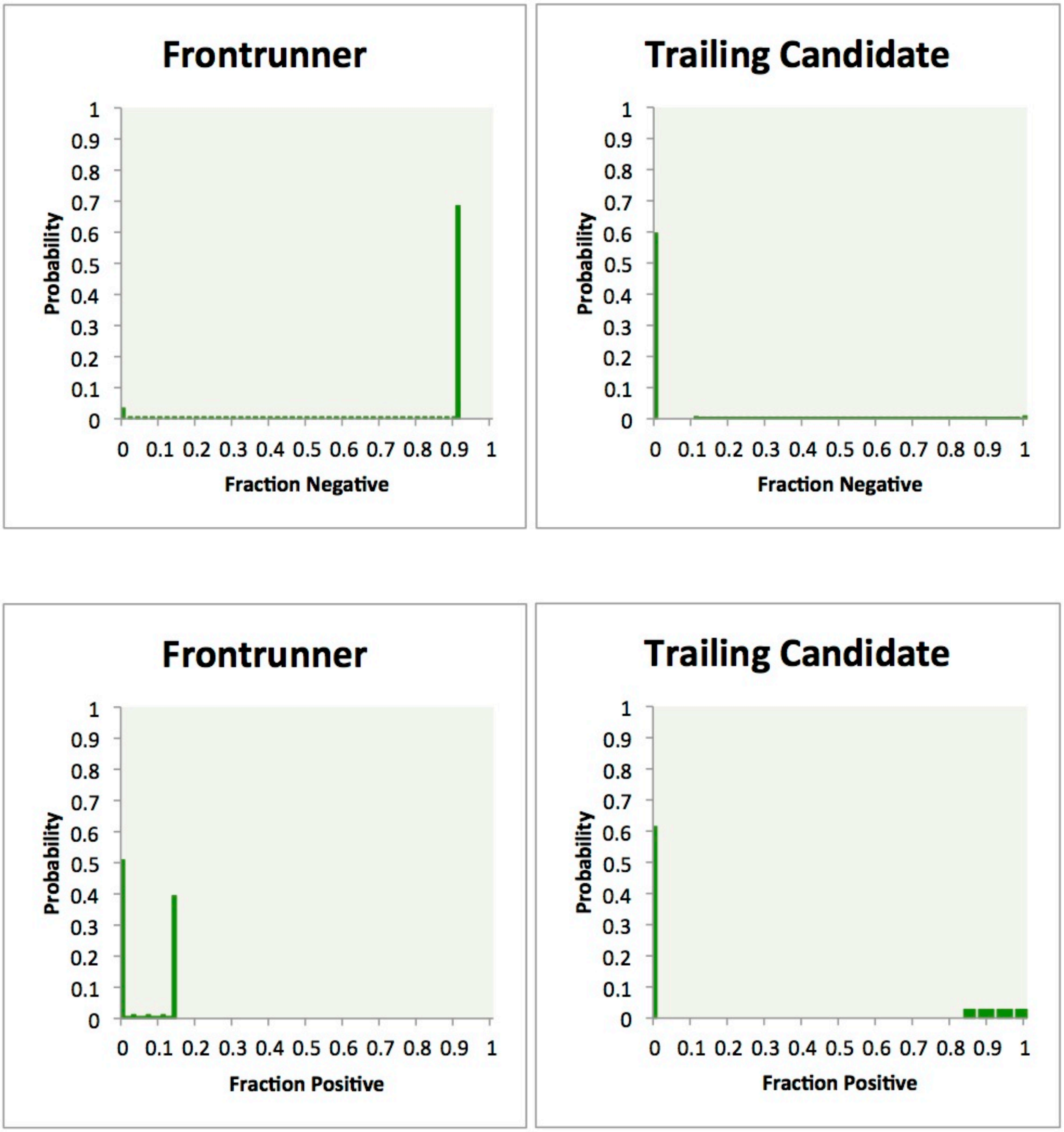
Note: Probability distributions for matrices sized 101x101. This example has G = 1, L = 2, B = 4.5, $X_F - X_T = 1$, $R_F = 5$, $R_T = 4$, D = 0.5.

Figure 4: Discrete equilibrium probability distributions, existence region (ii)



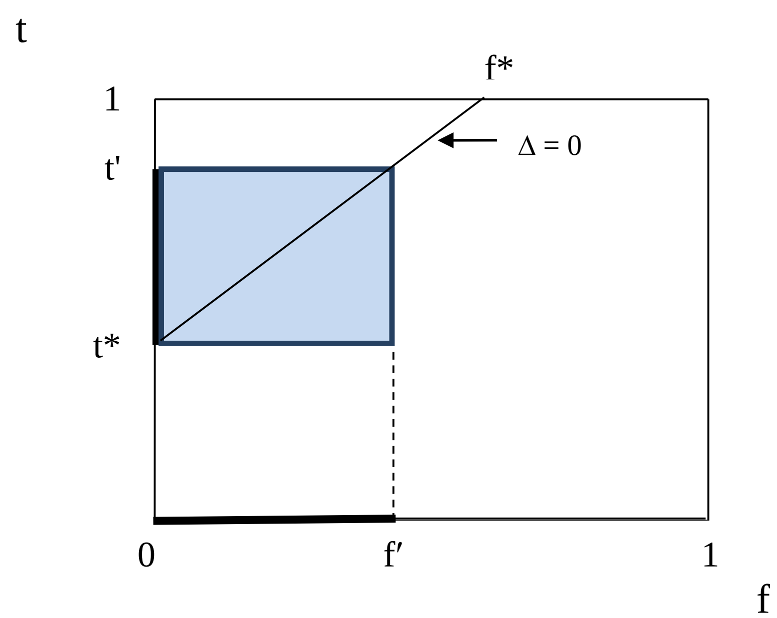
Note: Probability distribution is for a 101x101 matrix. This example has G = 1, L = 1.5, B = 15, $X_F - X_T = 2$, $R_F = 15$, $R_T = 12$, D = 1.

Figure 5: Discrete probability distributions, non-existence examples 1 and 2



Note: Probability distributions are for 101x101 matrices. The first example has G = 1, L = 1.1, B = 20, $X_F - X_T = 0.1$, $R_F = 10$, $R_T = 10$, D = -0.05. The second example has G = 1, L = 1.5, B = 75, $X_F - X_T = 2$, $R_F = 15$, $R_T = 12$, D = 3.

Figure 6
Determining each candidate's probability of winning the election



Appendix: Proofs

Proof of Lemma 1: Under assumptions (1) and (2), a pair (f, t) is not a pure strategy equilibrium either if (i) $\Delta(f, t) \neq 0$ and either $f > 0$ or $t > 0$ or (ii) $\Delta(f, t) = 0$ and either $f < 1$ or $t < 1$. If (i) holds and $f > 0$, then f could be reduced slightly, increasing θ^F but not changing $h(\Delta)$, thereby raising F's post-election reputation. A similar argument holds if $t > 0$. If (ii) holds and $f < 1$, an arbitrarily small increase in f would cause an infinitesimal drop in θ^F but would raise $h(\Delta)$ from ½ to 1, causing a discrete increase in F's post-election reputation. A similar argument holds in this case when $t < 1$. Conditions (a) and (b) are the only remaining possibilities. Q.E.D.

Proof of Lemma 2: For any t with $0 \leq t < t^*$ and any f, F wins the election so that $h(\Delta(f, t)) = 1$ and $U^T(t, f) = X_T + \theta^T(t, f)$. Then, for any $\Gamma^F(f)$, $EU^T(t, \Gamma^F) = X_T + E_f[\theta^T(t, f)]$ which is strictly decreasing in t. Hence, $EU^T(0, \Gamma^F) > EU^T(t, \Gamma^F)$, any t, $0 < t < t^*$. This means that candidate T will not put any probability weight in the interval $(0, t^*)$ making $\Gamma^T(t^* - \varepsilon) = \Gamma^T(0)$, for any $\varepsilon > 0$ as asserted in the Lemma. Similarly, for any f with $f^* < f \leq 1$ and any t, F wins so that $h(\Delta(f, t)) = 1$ and $U^F(f, t) = X_F + \theta^F(f, t) + B_F$. For any $\Gamma^T(t)$, $EU^F(f, \Gamma^T) = X_F + B_F + E_t[\theta^F(f, t)]$ which is strictly decreasing in f. Hence, $EU^F(f^*, \Gamma^T) > EU^F(f, \Gamma^T)$ so that F will put no probability weight on the interval $(f^*, 1]$ with $\Gamma^F(1) = \Gamma^F(f^*) = 1$ as asserted. Q.E.D.

Proof of Lemma 3: Assume that there exist a t' and an f' with $t' = m^T(f')$ and $\gamma^T(t') > 0$. Then $h(\Delta(f', t)) = 1$ if $t < t'$, $h(\Delta(f', t')) = ½$, and $h(\Delta(f', t)) = 0$ if $t > t'$. Hence, the expected payoff to F when playing f' is:

$$EU^F(f', \Gamma^T) = X_F + E_t[\theta^F(f', t)] + B_F(\Gamma^T(t') - \gamma^T(t')/2).$$

Consider F playing a strategy slightly more negative, f' + ε. Let $\delta = m^T(f' + \varepsilon) - t'$. Because a cdf has only a countable number of atoms, a sequence of ε going to 0 can always be chosen such that $\gamma^T(t' + \delta) = 0$ for every δ in the sequence. Then, F's expected utility at f' + ε is:

$$EU^F(f'+ \varepsilon, \Gamma^T) = X_F + E_t[\theta^F(f' + \varepsilon, t)] + B_F\Gamma^T(t' + \delta).$$

Since Γ^T is right hand continuous and θ^F is continuous, $\lim_{\varepsilon\to 0}[\Gamma^T(t' + \delta) - \Gamma^T(t')] = 0$ and $\lim_{\varepsilon\to 0}[E_t[\theta^F(f' + \varepsilon, t)] - E_t[\theta^F(f', t)]] = 0$. Hence, $\lim_{\varepsilon\to 0}[EU^F(f'+ \varepsilon, \Gamma^T) - EU^F(f', \Gamma^T)]$ $= B_F\gamma^T(t')/2 > 0$. Thus, F's expected payoff at f' is strictly less than at nearby f so that $\gamma^F(f') = 0$ must hold, yielding $\gamma^T(t')\gamma^F(f') = 0$ as asserted. Q.E.D.

Proof of Lemma 4:

(a) T's expected payoff at $t' = m^T(f')$ is:

$$EU^T(t', \Gamma^F) = X_T + E_f[\theta^T(t', f)] + B_F(\Gamma^F(f') - \gamma^F(f')/2).$$

Consider values of t = t' - δ, for values of δ > 0. At such values, T's expected payoff is:

$$EU^T(t' - \delta, \Gamma^F) = X_T + E_f[\theta^T(t' - \delta, f)] + B_T(\Gamma^F(f' - \varepsilon) - \gamma^F(f' - \varepsilon)/2)$$

where $f' - \varepsilon = m^T(t' - \delta)$.

Since Γ^F has at most a countable number of atoms, if $\gamma^F(f' - \varepsilon) > 0$ for some ε, then there are arbitrarily close values of t > t' - δ at which $\gamma^F(m^T(t)) = 0$. T's expected payoff at those values of t is strictly greater than at t' - δ. Hence, T puts no probability weight at any t' - δ at which $\gamma^F(m^F(t' - \delta)) > 0$. Then consider a t' - δ at which $\gamma^F(m^T(t' - \delta)) = 0$:

$$EU^T(t', \Gamma^F) - EU^T(t' - \delta, \Gamma^F) = E_f[\theta^T(t', f) - \theta^T(t' - \delta, f)] + B_T[\Gamma^F(m^T(t')) - \Gamma^F(m^T(t' - \delta)) - \gamma^F(m^T(t'))/2].$$

For any sequence of δ going to 0, $\theta^T(t' - \delta, f)$ converges to $\theta^T(t', f)$ since $\theta^T(t', f)$ is continuous. Since Γ^F is right hand continuous but has an atom at $f' = m^T(t')$, as δ converges to 0, $\Gamma^F(m^T(t')) - \Gamma^F(m^T(t' - \delta))$ converges to $\gamma^F(m^T(t'))$. Hence, $\lim_{\delta \to 0}[EU^T(t', \Gamma^F) - EU^T(t' - \delta, \Gamma^F)] = B_T\gamma^F(f')/2 > 0$. There must then exist some δ' such that $EU^T(t', \Gamma^F) > EU^T(t' - \delta, \Gamma^F)$, all $0 < \delta < \delta'$. T will put no probability weight in the interval [t' - δ', t']. Since $\gamma^T(t') = 0$ from Lemma 3, $\Gamma^T(t') = \Gamma^T(t' - \delta')$ as asserted.

(b) The proof follows identically to that in (a), switching the roles of F and T. Q.E.D.

Proof of Lemma 5: Consider any t with $m^T(b) > t > m^T(a)$. Then $EU^T(t, \Gamma^F) = X_T + E_f[\theta^T(t, f)] + B_T\Gamma^F(a)$. From (1), this is decreasing in t. Hence, T will put no probability weight on t in the given interval. Q.E.D.

Proof of Lemma 6: Assume that the Lemma is not true and that there exists an f′, $0 < f' < f^*$ with $\gamma^F(f') > 0$. Then from Lemma 3, $\gamma^T(m^T(f')) = 0$ must hold. From Lemma 4, $\Gamma^T(m^T(f')) = \Gamma^T(m^T(f' - \delta))$ for some $\delta > 0$. Then, $EU^F(f', \Gamma^T) = X_F + E_t[\theta^F(f', t)] + B_F\Gamma^T(m^T(f'))$ and $EU^F(f' - \varepsilon, \Gamma^T) = X_F + E_t[\theta^F(f' - \varepsilon, t)] + B_F\Gamma^T(m^T(f'))$. Taking their difference, $EU^F(f' - \varepsilon, \Gamma^T) - EU^F(f', \Gamma^T) = E_t(\theta^F(f' - \varepsilon, t) - \theta^F(f', t)] > 0$. This contradicts F being willing to put probability weight at f', so $\gamma^F(f') = 0$ must hold as asserted. A similar contradiction follows if $\gamma^T(t') > 0$, some $t^* < t' < 1$ is assumed. Q.E.D.

Proof of Lemma 7: Since there does not exist a pure strategy equilibrium, not all probability weight can be at an atom at 0. From Lemma 2, F must have probability weight in the interval (0, f*) and from Lemma 6, none of this is at an atom. Assume that there is an interval $(a, b) \in (0, f^*)$ with $\Gamma^F(b) = \Gamma^F(a)$ but $\Gamma^F(b + \delta) > \Gamma^F(b)$, for any small $\delta > 0$. From Lemma 5, $\Gamma^T(m^T(b)) = \Gamma^T(m^T(a))$ must also hold. Then:

$$EU^F(b + \delta, \Gamma^T) = X_F + E_t[\theta^F(b + \delta, t)] + B_F\Gamma^T(m^T(b + \delta)),$$

while at a,

$$EU^F(a, \Gamma^T) = X_F + E_t[\theta^F(a, t)] + B_F\Gamma^T(m^T(a)).$$

Since $\lim \Gamma^T(m^T(b + \delta)) = \Gamma^T(m^T(b)) = \Gamma^T(m^T(a))$, then, given (1):

$$\lim_{\delta \to 0}[EU^F(b + \delta, \Gamma^T) - EU^F(a, \Gamma^T)] = E_t[\theta^F(b, t)] - E_t[\theta^F(a, t)] < 0.$$

This contradicts F placing probability weight above b. There must thus exist an f′, $0 < f' \leq f^*$, with $\Gamma^F(f)$ strictly increasing between 0 and f′. Q.E.D.

Proof of Lemma 8: From Lemmas 6 and 7, $\Gamma^F(f)$ is continuous and increasing over (0, f′) and $\Gamma^T(t)$ is continuous and increasing over (t*, t′). Combining this with Lemma 4, F cannot have an atom at f′ and T cannot have an atom at t′. Thus, the only possible atom for F is at f = 0 and the only possible atoms for T are at t =0 or t = t*. The expected utility for T must be at least as high at $t^* + \varepsilon$ as at 0 since T puts probability weight near t*, from Lemma 7. Hence, for all $\varepsilon > 0$, $X_T + [E_{\tilde{f}}[\theta^T(t^* + \varepsilon, \tilde{f})] + B^T\Gamma^F(m^F(t^* + \varepsilon)) \geq$

$X_T + E_{\tilde{f}}[\theta^T(0, \tilde{f})]$. Since $\lim_{\delta \to 0} \Gamma^F(m^F(t^* + \varepsilon)) = \gamma^F(0)$, this yields the given bound on $\gamma^F(0)$. Since F has an atom at 0 then T cannot have one at t* from Lemma 3. Q.E.D.

Proof of Theorem 1: For candidate F to be willing to put probability weight in the interval [0, f'] consistent with Lemmas 7 and 8, EU^F must be constant on that interval. That is:

$$EU^F(f, \Gamma^T) = X_F + E_{\tilde{t}}[\theta^F(f, \tilde{t})] + B_F\Gamma^T(m^T(f)) = C_F,\ 0 \le f \le f'.$$

Solving yields:

$$\Gamma^T(m^T(f)) = [C_F - X_F - E_{\tilde{t}}[\theta^F(f, \tilde{t})]] / B_F,\ 0 \le f \le f'.$$

Substituting $t = m^T(f)$ and $f = m^F(t)$ yields:

$$\Gamma^T(t) = [C_F - X_F - E_{\tilde{t}}[\theta^F(m^F(t), \tilde{t})]] / B_F,\ t^* \le t \le t'.$$

Since $\Gamma^T(t') = 1$ from Lemma 7, $C_F = B_F + X_F + E_{\tilde{t}}[\theta^F(m^F(t'), \tilde{t})]$, which gives:

$$\Gamma^T(t) = 1 + [E_{\tilde{t}}[\theta^F(m^F(t'), \tilde{t})] - E_{\tilde{t}}[\theta^F(m^F(t), \tilde{t})]] / B_F,\ t^* \le t \le t'.$$

From Lemma 2, $\Gamma^T(t^* - \varepsilon) = \Gamma^T(0)$, all $0 < \varepsilon < t^*$. Since T can only have an atom at 0, $\gamma^T(0) = \Gamma^T(t^*)$.

For T to be willing to put probability everywhere in the interval [t*, t'], EU^T must be constant on that interval with:

$$EU^T(t, \Gamma^F) = X_T + E_{\tilde{f}}[\theta^T(t, \tilde{f})] + B_T\Gamma^F(m^F(t)) = C_T,\ t^* \le t \le t'.$$

Solving yields:

$$\Gamma^F(m^F(t)) = [C_T - X_T - E_{\tilde{f}}[\theta^T(t, \tilde{f})]] / B_T,\ t^* \le t \le t'.$$

Since Γ^T is constant on [t', 1], then from Lemma 5, Γ^F is constant on [f', 1] with f' = m^F(t'). Then Γ^F(f') = 1 must hold. Using this to solve for C_T and substituting into the expression for $\Gamma^F(m^F(t))$ yields:

$$\Gamma^F(m^F(t)) = 1 + [E_{\tilde{f}}[\theta^T(t', \tilde{f})] - E_{\tilde{f}}[\theta^T(t, \tilde{f})]] / B_T, \; t^* \leq t \leq t'.$$

To tie down the value of f' and t', three possibilities exist: T has an atom at 0, T has a higher expected utility anywhere in the range (t', t*) than at 0, or T is indifferent between 0 and points in the range (t', t*) but has no atom at 0. If T has an atom, then $\gamma^T(0) = \Gamma^T(t^*) = 1 + E_{\tilde{t}}[\theta^F(f', \tilde{t})] - E_{\tilde{t}}[\theta^F(0, \tilde{t})]] / B_F > 0$ or:

$$E_{\tilde{t}}[\theta^F(m^T(t', \tilde{t})] > E_{\tilde{t}}[\theta^F(0, \tilde{t})]] - B_F.$$

T must then be indifferent between t = 0 and any t with $t^* < t \leq t'$. Since T always wins when playing t' but never wins when playing 0, this implies that $E_{\tilde{f}}[\theta^T(t', \tilde{f})] = E_{\tilde{f}}[\theta^T(0, \tilde{f})] - B_T$. If T is not willing to have an atom at t = 0, $E_{\tilde{f}}[\theta^T(t', \tilde{f})] > E_{\tilde{f}}[\theta^T(0, \tilde{f})] - B_T$. Since t* is the lower bound on the support of T's distribution, and there is no atom there from Lemma 8, then $\Gamma^T(t^*) = 0$ must hold. From the formula for $\Gamma^T(t)$, $B_F + E_{\tilde{t}}[\theta^F(m^F(t'), \tilde{t})] - E_{\tilde{t}}[\theta^F(m^F(t^*), \tilde{t})] = 0$ or $E_{\tilde{t}}[\theta^F(f', \tilde{t})] = E_{\tilde{t}}[\theta^F(0, \tilde{t})] - B_F$. In the third case where T is indifferent but has no atom, these conditions, when combined with those in each of the first two cases, yield (a) and (b). Q.E.D.

Proof of Lemma 9: Substituting θ^I(i, j), m^T(f), and t* into Theorem 1 yields $\Gamma^F(f) = 1 + R_T[(t - t')(G + DR_F\mu_F)]/B$ and $\gamma^F(0) = 1 + R_T[(t^* - t')(G + DR_F\mu_F)]/B$. For any assumed value of μ_f, a cdf is determined whose mean can be computed by taking the derivative of $\Gamma^f(f)$ found after substituting $t = (R_F/R_T)f + t^*$. Then the mean of this cdf is

$\hat{\mu}_F = \gamma^F(0) \cdot 0 + \int_0^{f'} f(\frac{\mathrm{RF}}{B})(G + D\mathrm{RF}\mu_F)df = \frac{RF[G+DRF\mu_F]}{2B}(f')^2$. Setting $\hat{\mu}_F = \mu_F$ and solving yields the expression given for μ_F. Similarly, μ_T can be found from substituting into $\Gamma^T(t)$. Q.E.D.

Proof of Theorem 2: The general parameter restrictions in the Theorem are equivalent to conditions (7) – (10) with G set equal to 1 given the homogeneity of degree 0 in G, B, D, L, X_F, and X_T. These ensure that negative campaigning is always effective and rule out pure strategy equilibria. Substituting the values of μ_i given in Lemma 9 into the cdf's of Theorem 1 after having substituted the specific functions of the example yields the $\Gamma^F(f)$ and $\Gamma^T(t)$ given in (I) of the Theorem. For these to be valid either (a) or (b) of Theorem 1 must hold. In (a), T is willing to place probability weight on 0 so must receive the same expected payoff at 0 as at t′. Substituting $\theta^T(t, f)$ into the expected payoffs for T at these two values yields:

$$- GR_Tt' - DR_FR_Tt'\mu_F + B = 0$$

Since $B > GR_Tt'$ from (9), this can only hold if $D > 0$. Substituting the value of μ_F from Lemma 9 and simplifying yields the following quadratic equation in f′:

$$(f')^2 + (2G/(DR_F))f' - 2(B - GR_Tt^*)/(D(R_F)^2) = 0$$

Solving this using the quadratic formula yields:

$$f' = (R_T/R_F)(t' - t^*) = - G/(DR_F) + [(G/(DR_F))^2 + 2(B - GR_Tt^*)/(D(R_F)^2)]^{1/2} \quad \text{(A1)}$$

Of the two roots in the quadratic formula, only the + root is valid since $f' > 0$ must hold. In addition, for this solution to be valid, $f' \leq f^*$ must hold. This is true if and only if

$$B \leq (1/2)(D(R_T)^2)(1 - t^*)^2 + GR_T \quad \text{(A2)}$$

Also, at this solution, $0 \leq \gamma^T(0)$ must hold. Substituting the value for t' from (A1) into $\gamma^T(0) = \Gamma^T(t^*)$ and manipulating yields:

$$B \leq (3/2)G^2/D + GR_T t^* \quad \text{(A3)}$$

Thus, given the initial restrictions, $D > 0$, and (A2) and (A3) are necessary and sufficient for there to exist a t' with $t^* < t' \leq 1$ with $EU^T(0, \Gamma^F) = EU^T(t', \Gamma^F)$ and $0 \leq \gamma^T(0)$. Hence, Γ^F is a valid cdf and is the unique Nash equilibrium in this case. These conditions convert into those given in case (i). The lower bound on D follows from (A2) and the upper bound from (A3). The upper bound on B follows from the requirement that the upper bound on D not be smaller than the lower bound.

In (b) of Theorem 1, $\gamma^T(0) = 0$ holds. Since $\gamma^T(0) = \Gamma^T(t^*)$, this is true if

$$2B - D(R_T)^2[(t')^2 - (t^*)^2] + 2GR_T(t^* - t') = 0 \quad \text{(A4)}$$

As above, given (9) and $t^* < t'$, this is possible only if $D > 0$. Solving this quadratic equation yields:

$$t' = - G/(DR_T) + [(G/(DR_T) + t^*)^2 + 2B/(D(R_T)^2)]^{1/2} \quad \text{(A5)}$$

where again the positive root is taken to ensure that $t' > t^*$. Imposing $t' \leq 1$ yields the following parameter restriction:

$$B \leq (1/2)(D(R_T)^2(1-(t^*)^2) + (1-t^*)GR_T \qquad \text{(A6)}$$

T is willing to have $\gamma^T(0) = 0$ iff $EU^T(0, \Gamma^F) \leq EU^T(t', \Gamma^F)$. This holds iff:

$$0 \leq 2B - D(R_T)^2(t'-t^*)^2 - 2GR_T t' \qquad \text{(A7)}$$

Subtracting (A4) from (A7) yields:

$$t' \geq t^* + G/(D(R_T)^2) \qquad \text{(A8)}$$

Substituting (A5) for t' yields the following restriction on B for (A8) to hold:

$$(3/2)G^2/D + t^*GR_T \leq B \qquad \text{(A9)}$$

Thus, the parameter restrictions from (7) – (10), $D > 0$, and (A6) and (A9) are necessary and sufficient for (b) of Theorem 1 to hold for this specific case. (A6) and (A9) yield the upper bounds on D in (ii) of the theorem.

Since an equilibrium exists iff the conditions in (I) hold, no equilibrium exists if both (i) and (ii) are violated given (7) – (9). The condition in (iii) holds iff both (i) and (ii) fail. To show that, note that there are three ways to violate the conditions in both (i) and (ii):

(a) $D < \frac{2(B-GR_T)}{(R_T)^2(1-t^*)^2}$ $and\ D < \frac{3G^2}{2(B-GR_Tt^*)}$

(b) $D < \frac{2(B-GR_T)}{(R_T)^2(1-t^*)^2}$ $and\ D < \frac{2(B-GR_T(1-t^*))}{{R_T}^2(1-(t^*)^2)}$

(c) $D > \frac{3G^2}{2(B-GR_Tt^*)}$ $and\ D < \frac{2(B-GR_T(1-t^*))}{{R_T}^2(1-(t^*)^2)}$

Violations of the condition $B \leq \frac{GR_T}{2}(3-t^*)$ in (i) need not be considered directly since if that is violated, then $\frac{2(B-GR_T)}{(R_T)^2(1-t^*)^2} > \frac{3G^2}{2(B-GR_Tt^*)}$ and D must violate one of the other conditions in (i). Condition (b) subsumes conditions (a) and (c), yielding (iii). To see this, assume (a) holds but not (b). Then, $D < \frac{2(B-GR_T)}{(R_T)^2(1-t^*)^2}$, $D < \frac{3G^2}{2(B-GR_Tt^*)}$, and $\frac{2(B-GR_T(1-t^*))}{{R_T}^2(1-(t^*)^2)} \leq D$. For these to occur, $\frac{2(B-GR_T)}{(R_T)^2(1-t^*)^2} > \frac{2(B-GR_T(1-t^*))}{{R_T}^2(1-(t^*)^2)}$ must hold which implies that $B > \frac{GR_T}{2}(3-t^*)$ and $\frac{3G^2}{2(B-GR_Tt^*)} > \frac{2(B-GR_T(1-t^*))}{{R_T}^2(1-(t^*)^2)}$ must hold, which implies that $B < \frac{GR_T}{2}(3-t^*)$, a contradiction. Thus, if (a) holds so must (b). A similar argument shows that if (c) holds then so must (b). Q.E.D.

Proof of Theorem 3: First, consider the atoms at 0 negativity. Using the values of $\gamma^F(0)$ and $\gamma^T(0)$ given in Theorem 2 and the fact that $(\frac{R_F}{R_T})f' = t' - t^*$, algebraic manipulation yields that $\gamma^F(0) - \gamma^T(0) = D R_F (f')^2 t^* \alpha\beta$, which is always positive when an equilibrium exists.

Second, consider $\mu_F - \mu_T$. Substituting the expressions given in Lemma 9 and using $(\frac{R_F}{R_T}) f' = t' - t^*$ yields after algebraic manipulation:

$$\text{sign}[\mu_F - \mu_T] = \text{sign}[(R_T - R_F)(t' + t^*)(2B - D(R_T)^2(t' - t^*)^2) - 4BR_T t^*] \tag{A10}$$

From the inequality in (10) that $L(R_T - R_F) < X_F - X_T$, it follows that $R_T - R_F < R_T t^*$. The expression in (A10) is most likely to be positive when $R_T - R_F$ is at its upper bound. However, substituting $R_T t^*$ for $R_T - R_F$, (A10) becomes:

$$\text{sign}[\mu_F - \mu_T] = \text{sign}[(t' + t^* - 2)(2B) - (t' + t^*)D(R_T)^2(t' - t^*)^2)]$$

This is always negative. Hence, $\mu_T > \mu_F$ always holds. Q.E.D.

Proof of Theorem 4: To analyze which candidate wins, the outcomes in the strategy space (f, t) at which there is positive probability weight in equilibrium are shown in Figure 6. There are three such areas which are mutually exclusive and exhaust all probability weight:

(1) T plays at the atom at 0. This is the dark line on the horizontal axis between 0 and f'. There is probability weight of $\gamma^T(0)$ on this line which lies completely below the $\Delta = 0$ curve, so that F wins anywhere on the line.

(2) F plays at his atom at 0 but T plays on the continuous part of her distribution. This is the dark line on the vertical axis between t* and t'. The probability weight on this line is $\gamma^F(0)(1 - \gamma^T(0))$. The line is everywhere strictly above the $\Delta = 0$ curve except at the point where f = 0 and t = t* which is on the curve. Since T

does not have an atom at t*, that point arises with 0 probability. Hence, T wins the election everywhere on this line.

(3) Both F and T play on the continuous parts of their distributions. Since each has a uniform pdf, probability weight is spread uniformly over the shaded rectangle shown in the Figure, whose height is given by the interval [t*, t'] and whose base is given by (0, f']. The $\Delta = 0$ curve is the diagonal of this rectangle, with equal areas in the rectangle above and below it. Thus, in this circumstance, each candidate has the same probability of winning which equals $(1 - \gamma^F(0))(1 - \gamma^T(0))/2$.

Thus, $\pi_T = \gamma^F(0)(1 - \gamma^T(0)) + (1 - \gamma^F(0))(1 - \gamma^T(0))/2$ and $\pi_F = \gamma^T(0) + (1 - \gamma^F(0))(1 - \gamma^T(0))/2$. Then, taking the difference, $\pi_T - \pi_F = \gamma^F(0)(1 - \gamma^T(0)) - \gamma^T(0)$ as asserted in the Theorem.

In existence region (ii), $\gamma^T(0) = 0$ and $\pi_T > \pi_F$ follows immediately since $\gamma^F(0) > 0$ must hold. In existence region (ii), the following examples show cases where $\pi_T > \pi_F$ and $\pi_T < \pi_F$ hold respectively:

Example (a): L = 2, G = 1, B = 10, $R_F = R_T = 9$, D = 0.2, and X = 3.6

Example (b): L = 2, G = 1, B = 4.01, $R_F = R_T = 4$, D = 1, and X = 3

The results follow by substituting into the formulas for $\gamma^F(0)$ and $\gamma^T(0)$ given in Theorem 2. Note that $R_F = R_T$ was chosen to hold in both of these examples but this is of course not necessary. Q.E.D.

Proof of Results in Table 3: To find the comparative statics results given in Table 3, rewrite μ_F and μ_T as $\mu_T = \frac{G}{RTD(\frac{2BD}{A_T}-1)}$ and $\mu_T = \frac{G}{\mathrm{RF}D(\frac{2BD}{A_F}-1)}$ where $A_T = D^2(R_T)^2((t')^2 - (t^*)^2)$ and $A_F = (DR_F f')^2$. Expressions for t' and f' are given in Theorem 2 for each region. In each mean, some parameters only enter through A_T or A_F. For those parameters, only the derivative of A_T or A_F needs to be taken to determine the comparative statics sign. Where the sign is unambiguous, we can show analytically that it has the given value. When the sign is ambiguous (+/-), we have constructed examples consistent with the relevant region for which each sign occurs. These arguments and examples are available upon request. Q.E.D.

Proof of Theorem 5: Since F always has an atom at 0, $EU^F = E_tU^F(0, t)$. In existence region (ii), $\gamma^T(0) = 0$. Hence, F always loses the election when playing 0. Therefore, with G set at 1:

$$E_tU^F(0, t) = X_F + R_F - LR_T\mu_T = X_F + R_F - \frac{L}{D(\frac{2BD}{A_T}-1)}$$

where $A_T \equiv D^2(R_T)^2((t')^2 - (t^*)^2)$. Also, $\tau \equiv R_Tt^*$. Substituting the expression for t' from Theorem 2 yields:

$$A_T = \left[-1 + \sqrt{(1 + D\tau)^2 + 2BD}\right]^2 - (D\tau)^2$$

The variables X_F, X_T, R_F, and R_T only enter A_T through τ. Hence,

$$dEU^F/dX_F = 1 - \frac{2BDL}{D(\frac{2BD}{A_T}-1)^2(A_T)^2}\frac{dA_T}{d\tau}\frac{d\tau}{dX_F} \text{ where:}$$

$$\frac{dA_T}{d\tau} = [2D(1 + D\tau)\frac{-1+\sqrt{(1+D\tau)^2+2BD}}{\sqrt{(1+D\tau)^2+2BD}} - 2D^2\tau > 0$$

and $\frac{d\tau}{dX_F} = \frac{1}{L-1}$. The following example is consistent with the restrictions for region (ii) and yields a negative value for dEU^F/dX_F :

G = 1, L = 2, B = 25, $R_F = R_T = 24$, D = 0.08, and $X_F - X_T = 0.01$

dEU^F/dR_F is the same as dEU^F/dX_F. X_T and R_T enter identically only through τ in μ_T. Since $dA_T/d\tau$ is always positive and $d\tau/dX_T$ is negative, an increase in X_T or R_T always raises EU^F.

Now consider EU^T in existence region (i). Since $\gamma^T(0) > 0$ and T always loses when playing 0:

$$E_fU^T(0, f) = X_T + R_T - LR_F\mu_F = X_T + R_T - \frac{L}{D(\frac{2BD}{A_F}-1)}$$

where $A_F = (DR_F f')^2$. Substituting for f' from Theorem 2 yields:

$$A_F = [-1 + \sqrt{1 + 2D(B-\tau)}]^2$$

Then

$$dEU^T/dX_T = 1 - \frac{2BDL}{D(\frac{2BD}{A_F}-1)^2 (A_F)^2} \frac{dA_F}{d\tau} \frac{d\tau}{dX_T}$$

and

$$\frac{dA_F}{d\tau} = \frac{-2DA_F}{\sqrt{1+2D}(B-\tau)} < 0$$

The following example is consistent with region (i) and yields a negative value for dEU^T/dX_T:

G = 1, L = 2, B = 25, $R_F = R_T = 24$, D = 0.04, and $X_F - X_T = 0.01$

As for EU^F, X_F and R_F only enter EU^T through τ in μ_F and thus an increase in them always causes an increase in EU^T. Q.E.D.