Private-Information Group Contests*

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Abstract

We model competing groups when players' values for winning are private information, each group's performance equals the best effort ("best shot") of its members, and the group with the best performance wins the contest. At the symmetric equilibrium, the overall expected best shot unambiguously increases with the number of competing teams, though each team's performance may increase or decrease. Depending on the convexity of the distribution of players' values, individual, team, and contest performance may increase or decrease with team size. Considering just two competing groups that differ in size, we show members of the smaller group use the more aggressive strategy. Nevertheless, depending on the nature of uncertainty, either team may be more likely to win.

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1 Introduction

Group competition abounds. It can be seen in trade association advertising (buy *California* wine!), industry groups lobbying for weaker pollution standards while environmental groups lobby for stricter regulation, pharmaceutical research labs competing to find a revolutionary weight-loss drug, and virtually any team sporting event. In these and many other instances, an individual exerts effort to help his team win, and members of the winning group enjoy the victory, perhaps with little regard to their individual efforts. Thus, a group member faces a dilemma: should he increase his effort in hopes of increasing his team's chance of winning, or should he reduce his efforts to save on personal costs and thereby free-ride on teammates?

We model group competitions as contests and examine how individual incentives depend on group size and the number of competing groups. Our primary innovation is to address these issues in a model of private information, so motivations of others—teammates and rivals—are only imperfectly known. To our knowledge, this is one of only two papers to study group contests with private information. The other is by Brookins and Ryvkin (2014), who take a different tack. Unlike their approach, ours admits closed-form solutions of equilibria (see ft. 9 below).

The study of group contests is a natural extension of the well-developed models of individualistic contests, where, given players' efforts, an individual winner is determined according to a contest success function (csfs). According to "imperfectly discriminating" csfs, efforts influence the probability that a player might win, but they generally are not determinative.[1] Of these, the most frequently used is Tullock's "lottery" contest, where one player's chance of winning simply equals that player's own effort relative to the total effort exerted in the contest. Alternative models, also known as "all-pay auctions," developed using so-called "perfectly discriminating" csfs, where the winning player is simply the one who exerts the greatest effort.[2]

Not surprisingly, given the prevalence of group competition in the real world, theorists readily adapted the individualistic contests to study groups. Two strains can be distinguished, depending on whether the contest prize is a private good or a public good. Where the prize is a private good, researchers have studied inter-group and intra-group incentives; for example, how do the rules for allocating the prize influence effort choices at the inter-group competition stage?[3] More relevant for our study is the second strain.[4] While larger groups may have more resources at their disposal, they also face greater incentives for free riding, leading Olson (1965) to conjecture that larger groups would perform more poorly in the provision of public goods. This "group-size paradox" has also been the subject of study in group contests. Here the seminal paper is

[1]Among others, see Tullock (1980), Dixit (1987), and Hirshleifer (1989).

[2]See, e.g., Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996).

[3]See, for example, Nitzan (1991a, 1991b), Lee (1995), Katz and Tokatlidu (1996), Wärneryd (1998), Münster (2007), and Choi, Chowdhury, and Kim (2013).

[4]See, for example, Katz, Nitzan, and Rosenberg (1990); Baik (1993, 2008); Baik, Kim, and Na (2001); Barbieri, Malueg, and Topolyan (2013); Chowdhury, Lee, and Sheremeta (2013a); and Kolmar and Rommeswinkel (2013)

that of Katz, Nitzan, and Rosenberg (1990), who found a team's effort did not depend on the number of players per team; and the team whose members had the larger value for winning was the team more likely to win. This lack of dependence on team size hinged on the assumption of constant marginal cost. Allowing for increasing marginal cost of effort, Riaz, Shogren, and Johnson (1995) and Esteban and Ray (2001) found team size matters: while increasing team size may reduce individual effort, it increases total team effort.

All of the foregoing group-competition models assume players have perfect information. The essential focus of our inquiry is on how private information affects individual incentives and equilibrium outcomes. The literature provides numerous examples of group conflict, e.g., matches between sports teams, rivalries among military alliances, and competitions involving R&D consortia or legal teams (see also footnote 4). For most of these environments assuming complete information is a useful modeling device, but considering private information is certainly more realistic. Moreover, private information allows one to model a richer set of behavioral responses. For instance, a change in incentives may induce the same agent to reduce his effort if his value is low, but to increase his contribution otherwise. Incorporating such complexity into complete-information models would require positing asymmetries among agents, quickly making the analysis intractable. Finally, a treatment of private information is an essential first step toward understanding issues such as information acquisition and dissemination and contest design for groups.[5]

We make several key assumptions. *First*, the prize is a pure public good for members of the winning team.[6] *Second*, each player's value for the prize is privately known to that player, though *ex ante* players know these values are independently drawn from a common distribution. *Third*, each group's performance equals the group's "best shot," that is, the largest effort exerted by one of the group's members.[7] *Fourth*, the winning group is determined by the perfectly discriminating csf selecting the group with the best performance. While only members of the winning group enjoy their private benefits, winners and losers alike incur their effort costs. The first, second, and fourth assumptions can commonly be found in models of individualistic or group contests. However, the third deserves comment.

One might suggest, instead, modelling group performance as a sum of efforts, as is usually done in the literature on private contributions to public goods[8] and in most of the group-contest models mentioned above. However, in a private-information setting this is wholly intractable, as a player must then consider both the whole *distribution* of the sum of contributions of his teammates *and* the distributions of rival teams'

[5] For a recent survey of these issues, see Konrad (2009).

[6] Our framework readily accommodates a prize for the "overall" winning player, over and above the team's, as long as the prize amount does not depend on efforts.

[7] Hirshleifer (1983) proposed several possibilities for aggregating group efforts: the best shot (equal to the maximum effort), the weakest link (equal to the minimum effort), and the summation. With full information, Kolmar and Rommeswinkel (2013) use a generalized group aggregation that allows for different degrees of complementarity among members' efforts in each group, covering the gamut from the weakest link to the summation, but not from summation to the best shot.

[8] The classic reference here is Bergstrom, Blume, and Varian (1986).

total efforts.[9] As we will see, the private-information best-shot model is tractable, requiring only the analysis of an order statistic.[10] Furthermore, the best-shot performance seems a better description than summation for some group contest situations, such as those involving the competition of "great ideas."

For instance, Morgan and Wang (2010, p. 79) distinguish *revolutionary* and *evolutionary* ideas as follows: "The probability of arriving at a revolutionary idea therefore hinges on the peak level of efforts by the contestants... Successful evolutionary ideas are therefore more dependent on aggregating a large pool of contestant idea contributions and effort." Similarly, Levitt (1995, p. 744) finds that "In many real-world situations, however, the principal's payoff is based solely on the 'best' of the agents' outputs (e.g., the first agent to make an innovation, the most creative advertising campaign, or the cheapest product design)." This is often reflected in the internal organization of a group. Indeed, Levitt (1995, p. 745) writes, "In the real world, multiple agents are often assigned to the same task even though *ex post* only one of the outputs is used. In the development of a new advertising campaign, as many as ten teams within an advertising agency will generate three potential campaigns each. Of these thirty ideas, only one or two will be presented to the client."

Also capturing well our assumptions are architectural design competitions, which are, for example, described in The Handbook of Architectural Design Competitions (THADC).[11] Especially related to contests for revolutionary ideas are "idea competitions."[12] Furthermore, "Design competitions are a search for the best... When properly run, design competitions elevate the level of public expectation for design excellence" (THADC, p. 40). Therefore, the use of the best-shot aggregator within and across teams appears justified by the highly creative nature of the work involved. The public-good nature of the prize arises from the individual and firm reputation benefits for the winner. Indeed, "The architect who is awarded first prize in a design competition for a project not only may win a commission for a project but public and professional recognition as well... American designers such as Kallmann, McKinnel & Wood; Mitchell/Giurgola; Venturi, Rauch, and Scott Brown; Geddes, Brecher, Qualls, Cunningham; Steven Holl; and Daniel Libeskind have all advanced themselves through their competition work" (THADC, p. 40). Additionally, owing to profit-sharing agreements that often exist among partners in a firm, individuals may benefit from a win even when their contributions are nil. Finally, "The classic competition is done in a single stage" (THADC p.

[9] Brookins and Ryvkin (2014) study an imperfectly discriminating contest among symmetric groups where players costs of effort are private information. Each team's effective effort equals the sum of its members efforts. They establish existence of a symmetric equilibrium under quite general conditions. Assuming a lottery csf, they depict equilibrium strategies using numerical techniques, and then with only two or three players or teams. Furthermore, they write, "Needless to say, there is no closed-form characterization of equilibrium bidding functions under incomplete information with continuously distributed types even for the simplest parameterizations" (p. 11).

[10] This simplification is also key to the analysis of private contributions to public goods in Barbieri and Malueg (2014).

[11] We thank Chris Ellis for this suggestion.

[12] Idea competitions are described as "These competitions are held for projects that are not intended to be built. They are useful as explorations of significant design issues but are limited insofar as they stop short of realization. Nevertheless, idea competitions can stimulate interest in untried possibilities" (THADC, p. 28).

14); "one-stage competitions select a winner and rank other prize-winning designs in a single sequence. The majority of design competitions are held in one stage" (THADC p. 29), and we too formulate the contest as one-stage.

Most closely related to the current paper is the perfect-information model of Barbieri, Malueg, and Topolyan (2013), which also models group contests as best-shot all-pay auctions.[13] Barbieri *et al.* (2013) show that in the symmetric equilibrium, an increase in team size leads to lower individual and group efforts and greater payoffs. Thus, as teams symmetrically increase in size, incentives to free-ride on teammates dominate; and this symmetric reduction in efforts benefits all players. Considering a contest between two teams of different size, but having common value for the prize, they show that the smaller team has the advantage—it is more likely to win. Moreover, increasing the disparity in team size enhances this advantage. In the current paper, we show these conclusions may qualitatively change with the introduction of private information.

We begin by studying *ex ante* symmetric teams. First, we characterize semi-symmetric equilibria,[14] finding in some cases the only such equilibrium has all players active (though we do give a sufficient condition for there to exist equilibria in which some players remain inactive). Focussing on the symmetric equilibrium, we then study the effects of increasing the number of teams, N, or the number of (active) members per team n.

The first basic economic force we analyze is the contest's competitiveness as reflected in N, the number of competing teams. We find an increase in N decreases players' payoffs but can increase or decrease *team* performance; overall, however, the increased competition increases *contest* performance (the expected value of the contest best effort). This contrasts with the individualistic contest, in which increased competitiveness can reduce individual performance so much that overall performance declines.[15] The difference arises because teams suffer from free riding: the addition of an extra team is not as detrimental to existing players because the new team suffers from internal free riding that limits its aggressiveness.

Two additional basic economics forces arise with a change in the number of players per teams. Increasing the common number of players per team, n, is less clearly an increase in the competitiveness of the contest:

[13]However, in that paper, all members on a team have the same value for winning; and in the symmetric model, this value is common across teams. Equilibrium is in mixed strategies.

Two other perfect-information public-good models of group contests should be mentioned. Chowdhury *et al.* (2013a) suppose a group's performance is given by the team's best shot, and the winning team is determined by a lottery csf taking as inputs the best-shots of each team. Baik *et al.* (2001) suppose each team's performance equals the sum of members' efforts and the winning team is the one with the largest total effort (thus, an all-pay auction). These authors find that in equilibrium only one player on a team exerts effort (for Baik *et al.* it is necessarily the player with the greatest value). Thus, these frameworks are not especially conducive to examining the role of team size or the number of teams in the competition.

[14]Semi-symmetric equilibria are those having a common number of active players per team (but not all players need be active), all of whom use the same strategy.

[15]"Theory suggests that *limiting* competition is often an effective strategy. By restricting entry, each contestant perceives a higher chance of winning the contest and raises effort accordingly. While the total 'bandwidth' devoted to the problem may be lower, peak bandwidth is higher and hence truly revolutionary ideas are more likely to be generated" (Morgan and Wang, 2010, p. 80).

first, there are more “other” players against whom one competes, which may lead individuals to work harder; but, second, there are also more players on one’s own team, which may encourage free riding. The nature of private information is critical in determining which of these two forces prevails. We show that if the distribution of players’ private values, F, is convex, then increasing n decreases individual, team, and contest performance—at the same time increasing payoffs. In this situation, as n increases, the inducement to greater free-riding dominates; and the lessening of competition generates greater payoffs. These effects track those of the perfect-information model. However, when F is concave, the effects can be reversed; specific effects may depend on a player’s value, with low-value players reducing their efforts while high-value players increase theirs. Intuitively, when F is concave high values are less and less likely. The addition of one teammate then has little effect on a high-value player, who expects a minor change to his chance of being the team’s best shot. Therefore, internal free-riding is limited and teams’ best efforts tend to increase because of a mechanical “order-statistic” effect (or “sampling” effect, in the words of Levitt, 1995). Thus, regarding efforts, an elegant symmetry exists: if F is convex (concave), then the teams’ and the overall best-shot efforts increase (decrease) with n.[16]

This symmetry does not extend to payoffs: as described above, *ex ante* expected payoffs increase with n for convex F; however, they do not necessarily decrease for concave F. This is due to a fourth and final basic economic force: because the winning team is determined by the best shot and an agent may win by taking advantage of a teammate’s effort, the allocation rule is not generally “efficient”; it does not necessarily assign the prize to the group with the largest realized *sum* of values, despite strategies being strictly increasing in values. It is then important to track the correlation between an agent’s value and his *interim* probability of winning—on average, agents benefit if the mechanism assigns them the prize only when their value is high. We show this correlation is decreasing in n, as an agent’s victory is more likely to be determined by a teammate. When F is convex, the reduction in expected effort is sure to dominate the reduced correlation effect. However, If F is only “slightly” concave, the cost savings from reduced efforts dominates, and payoffs increase with team size. But as F becomes more concave, the reduction in effort is less and the decreased correlation effect comes to dominate. Thus, if F is sufficiently concave, then individual payoffs may fall as team size increases. Interestingly, the role of the correlation effect is also highlighted by the possibility that *ex ante* utility may fall even when individual expected efforts fall.

Finally, we study players’ behavior when teams differ in size, thereby linking to the group-size paradox already highlighted. Considering two teams, we find that members of the smaller team are sure to use more aggressive strategies; thus, free riding is arguably greater on the larger team. However, because team performance equals the best effort, the larger team has an order-statistic advantage. The net effect of these

[16] This requires the additional assumption that the lowest possible value is zero.

two forces appears ambiguous. Indeed, we show in a family of examples that our earlier results on convexity carry through so either team may be more likely to win the contest, and when the smaller (larger) team has the advantage, increasing the size disparity enhances this advantage.

The rest of the paper is organized as follows. Section 2 describes the basic model. Section 3 analyzes contests among groups of the same size, exploring comparative statics results regarding the effects of team size and number on individual, team, and overall efforts. Section 4 analyzes contests between two groups of different size. Section 5 concludes. Most proofs are in the Appendix.

2 The model

We model a contest among N teams, each of which has n members. All players are *ex ante* symmetric, with individual values of winning being independently and identically distributed according to the atomless cumulative distribution function (cdf) F having support $[\underline{v}, \bar{v}]$ and probability density function (pdf) f, where $\bar{v} > \underline{v} \geq 0$. A player's realized value is private information for that player. So, while teams are *ex ante* symmetric, in the realized play of the game values will differ across and within teams. Players independently and simultaneously exert effort on behalf of their own teams. A player with value v exerting effort x has final payoff $v - x$ if his team wins and payoff $-x$ if his team loses. The winning team is determined as the team having the largest "best shot," where each team's best shot simply equals the greatest effort exerted by a team member.[17] The foregoing description is common knowledge.

3 Contests among groups of equal size

3.1 General characterizations

Active players are those who exert positive effort with strictly positive probability. We use the term *semi-symmetric equilibrium* to refer to an equilibrium in which active players who are in *ex ante* identical situations use the same strategies. Here we consider teams of equal size, initially taking all players to be active, in which case a semi-symmetric equilibrium is symmetric in the usual sense. Let g be the common symmetric equilibrium strategy. Standard arguments establish that g is continuous and increasing. Because a player with value $\underline{v}$ is sure all other players will exert greater effort than he, he chooses to exert zero effort. We record these properties as follows.

[17] Equivalently, the winning team is the one with the individual exerting the greatest effort among all players. In the case of ties for the overall best shot, the tying teams each have equal probabilities of being designated the winner. In equilibrium the chance of ties is zero.

Lemma 1. *In a symmetric equilibrium where all players use strategy g, it must be the case that g is continuous and strictly increasing, starting from 0.*

We now derive the symmetric equilibrium strategy g. Let F^M $\left(= (F)^{Nn-1}\right)$ be the cdf of the maximum of $Nn-1$ draws from F, and let f^M be the associated pdf. Acting like a type v' player, a player exerts effort $g(v'; n, N)$ and can win if either (i) all other players have value less than v', which happens with probability $F^M(v')$, or (ii) the largest of the other players' values exceeds v', which happens with probability $1 - F^M(v')$—and, in this case, the probability that a teammate has this largest value is $\frac{n-1}{nN-1}$ since players behave symmetrically. The player's corresponding interim probability of winning when he acts as if his value were v' is

$$\begin{aligned} p(v'; n, N) &\equiv F^M(v') + \frac{n-1}{nN-1}(1 - F^M(v')) \\ &= \frac{n-1}{nN-1} + \frac{n(N-1)}{nN-1} \left(F(v')\right)^{nN-1}. \end{aligned} \quad (1)$$

Consequently, a player with value v acting as if his value were v' has payoff[18]

$$U(v', v) = -g(v'; n, N) + v \cdot p(v'; n, N). \quad (2)$$

The first-order condition for the choice of v' is

$$0 = \frac{\partial U(v', v)}{\partial v'} = -g'(v') + v \frac{n(N-1)}{nN-1} f^M(v'); \quad (3)$$

equilibrium requires a player want to act like his true type ($v' = v$), so

$$g'(v) = \frac{n(N-1)}{nN-1} v f^M(v). \quad (4)$$

The differential equation (4) and the boundary condition $g(\underline{v}) = 0$ yield

$$g(v) = \frac{n(N-1)}{nN-1} \int_{\underline{v}}^{v} x f^M(x)\, dx = n(N-1) \int_{\underline{v}}^{v} x(F(x))^{Nn-2} f(x)\, dx. \quad (5)$$

For $n = 1$, (5) agrees with the standard all-pay auction equilibrium (see §5 of Hillman and Riley, 1989).

Proposition 1 (Semi-symmetric equilibria)**.** *(i) There exists a symmetric equilibrium in which all players follow the strategy given in* (5)*. (ii) If there exists $\varepsilon > 0$ such that $F(v) \geq v/\bar{v}$ for all $v \in (\underline{v}, \underline{v} + \varepsilon)$, with*

[18]Here we specify the full arguments of g as the value, the number of active players per team, and the number of teams. To simplify notation, we sometimes include, for g and other functions, only the variables of interest, as others are being held fixed.

strict inequality holding for some $v' \in (\underline{v}, \underline{v}+\varepsilon)$*, then this is the only semi-symmetric equilibrium. (iii) If* F *first-order stochastically dominates the uniform distribution on* $[0,\bar{v}]$*, then for any* $m \in \{1,\ldots,n\}$ *there also exists a semi-symmetric equilibrium in which exactly* m *players on each team are active, each using the strategy*

$$g(v;m,N) = m(N-1)\int_{\underline{v}}^{v} x(F(x))^{Nm-2} f(x)\,dx. \tag{6}$$

Proposition 1 provides the first indication that the nature of the cdf F will play an important role. On the one hand, as discussed below, convexity of F tends to promote free riding, and this is reflected in Proposition 1 by the presence of semi-symmetric equilibria in which some players are inactive. This accords nicely with the case of full information studied by Barbieri *et al.* (2013). There players had a common value, say $\bar{v}$, for winning and semi-symmetric equilibria had m players following a common mixed strategy while all other players remained inactive. If we consider the cdf $F(v) = (v/\bar{v})^t$ on $[0,\bar{v}]$, for $t > 1$, then the cdf is convex and for very large t closely approximates the case of complete information (see the example of Section 3.2). Thus, for large t, we expect equilibrium with inactive players to be possible; Proposition 1 shows this possibility extends at least to all convex cdfs.[19] On the other hand, we see that if F is strictly concave on $[0,\bar{v}]$, then such equilibria with inactive players do not arise—complete free riding is unattractive because low values for teammates are relatively likely.[20]

A question of primary importance is how competitiveness of the contest (N) and team size (n) affect players' utility and individual, team, and contest performance. As the number of teams increases, a player faces a more competitive contest, and we will show his interim payoff falls. The performance of individual teams may rise or fall with an increase in N, but the overall effect is to increase the contest performance, i.e., the overall expected best effort. Next we turn to the effects of team size. Because one may view the effort of a player as contributing toward the common good of his team's winning, the theory of private provision of public goods suggests the conjecture that players will exert less effort as the size of their teams increases. Regarding utility, if the equilibrium strategy g is played with n active players per team, then, from (2), the interim payoff of an active player with value v is

$$U^*(v;n,N) = -g(v;n,N) + v \cdot p(v;n,N). \tag{7}$$

Another natural conjecture is that increased free riding, which induces lower efforts, will yield higher payoffs. The following result establishes that, for convex F, these conjectures are correct: increasing team size decreases individual, team, and overall contest performances while increasing individual expected utility.

[19] This follows because a nonatomic convex cdf first-order stochastically dominates the uniform distribution on $[0,\bar{v}]$. Convexity is sufficient but not necessary for there to exist equilibria in which some players remain inactive.

[20] Clearly, F being strictly concave is sufficient but not necessary to fulfill the condition of part (ii) of Proposition 1.

For simplicity, the next theorem focusses on symmetric equilibria, which are sure to exist. In those cases where other semi-symmetric equilibria also exist, the following theorem may be interpreted, for example, as describing the effects (on active players) of a change from m to $m+1$ active players.

Theorem 1 (Effects of increasing group size or the number of competing groups)**.** *Consider the symmetric equilibrium in which N teams consist of n players each, each playing the strategy* (5)*.*

1. *Increasing the number, N, of competing n-player teams*

   (a) *decreases interim payoffs, that is, $U^*(v; N+1) < U^*(v; N)$ for any $v \in (\underline{v}, \bar{v})$;*

   (b) *increases the contest's overall expected best effort if $n \geq 2$; and*

   (c) *reduces each team's expected best effort if F is convex.*

2. *If F is convex, then increasing the number, n, of players per team*

   (a) *implies $g(v; n) > g(v; n+1)$ for every $v \in (\underline{v}, \bar{v})$;*

   (b) *decreases each team's expected best effort;*

   (c) *decreases the contest's overall expected best effort; and*

   (d) *implies $U^*(v; n) < U^*(v; n+1)$ for every $v \in (\underline{v}, \bar{v})$.*

Intuitively, an increase in N increases the contest's competitiveness, and this decreases players' payoffs. The increased competition tends to decrease team performance but does increase overall contest performance. This contrasts with the individualistic contest, in which increased competitiveness can reduce individual performance so much that overall performance actually decreases.[21] The difference arises because teams suffer from free riding: the addition of an extra team is not as detrimental to existing players because the new team suffers from internal free riding that limits its aggressiveness.

Theorem 1 yields the following corollary regarding individual players.

Corollary 1. *Consider the symmetric equilibrium in which N teams have n players each, each playing the strategy* (5)*. If F is convex, then increasing n reduces each player's expected effort and increases his* ex ante *expected utility.*

Corollary 1 accords with the symmetric full-information model of Barbieri *et al.* (2013), which showed in the symmetric mixed-strategy equilibrium that increasing the number of players per team lowers individual expected efforts and raises individual expected utility. More broadly, the finding is valid for all convex F.

[21] For example, if $n = 1$ and $F(v) = v^t$, where $v \in [0, 1]$, then the expected best effort of the contest decreases with N if $t > 5.4$.

Increasing the common number of players per team, n, has two countervailing effects on team performance: increased free-riding and a mechanical "order-statistic" effect of a more favorable distribution of a team's largest value. The relationship between these two effects is subtle, and this is why Theorem 1 gives only a sufficient condition involving convexity to resolve this ambiguity in favor of free-riding. The following proposition establishes that, with the additional assumption that $\underline{v} = 0$, the team and contest performances depend critically on the nature of the cdf of values, showing that increases in team size may increase or decrease a team's expected best-shot effort as the cdf is concave or convex.

Proposition 2 (Effects of increasing group size when F is convex or concave)**.** *Suppose F has support $[0, \bar{v}]$, and consider the symmetric equilibrium in which N teams have n players each, each playing the strategy* (5)*. Increasing the number of team members n*

1. *decreases each team's expected best effort and the contest's expected best effort if F is convex;*
2. *increases each team's expected best effort and the contest's expected best effort if F is concave; and*
3. *has no effect on each team's expected best effort or the contest's expected best effort F is the uniform distribution.*

Theorem 1 already treated the case of convex F. When F is concave (i.e., the pdf is decreasing), high values are relatively unlikely. Therefore, the addition of one teammate has little effect on a high-value player, who expects only a minor change to his chance of being his team's best shot. Consequently, within-team free-riding is limited, and teams' best efforts tend to increase, because of the more favorable distribution of the largest team value.

The elegant symmetry of Proposition 2 does not extend to *individual* expected efforts or *ex ante* utility. Indeed, as illustrated in the next subsection, when F is concave individual expected efforts may increase or decrease with team size. By analogy with the case of convex F, one might expect that, when individual expected efforts fall with n, expected utility will increase. However, *ex ante* utility can fall with an increase in team size, even when individual expected efforts fall. This occurs because it is not simply a team's *ex ante* probability of winning that matters in determining payoffs; rather, the correlation, $\operatorname{corr}\{v, p(v;n)\}$, of a player's value with his (interim) probability of winning is crucial. We can rewrite expected utility as

$$\begin{aligned}
\mathrm{E}[\, U^*(v) \,|\, n \,] &= -\mathrm{E}[\, g(v;n) \,] + \mathrm{E}[\, vp(v;n) \,] \\
&= -\mathrm{E}[\, g(v;n) \,] + \operatorname{corr}\{v, p(v;n)\} + \mathrm{E}[\, v \,]\, \mathrm{E}[\, p(v;n) \,] \\
&= -\mathrm{E}[\, g(v;n) \,] + \operatorname{corr}\{v, p(v;n)\} + \frac{1}{N}\mathrm{E}[\, v \,]\,. \qquad (8)
\end{aligned}$$

It is of course beneficial to have this correlation be strong; even if a team's *ex ante* chance of winning ($1/N$) is unchanged, a player's payoff will increase if his win is more likely when his value is large. Lemma 2 shows this correlation decreases with team size.

Lemma 2. *For any $n \geq 2$, $corr\{v, p(v; n)\}$ decreases as n increases*

Intuitively, as team size increases, a player's win is more likely to be due to a teammate's effort, and this weakens the link between the first player's value and his interim probability of winning. Lemma 2 and equation (8) imply that a necessary condition for an increase in team size to increase payoffs is that individual expected efforts fall. The example of the following subsection, based on power function distributions of values, examines closed-form equilibrium strategies, sharpening and extending the results obtained so far.

3.2 A family of examples: the power function distribution

Throughout this section we assume $F(v) = v^t$ on $[0, 1]$, where $t > 0$ is a parameter, and we focus on the equilibrium in which all players are active.[22] Given N groups with n members per team, equation (5) yields

$$g(v) = \left(\frac{(N-1)nt}{Nnt + 1 - t} \right) v^{Nnt+1-t}. \tag{9}$$

If $t \geq 1$, then by Theorem 1 a player's strategy g shifts down as n increases. This may be viewed as increased free-riding as teams get larger. In contrast, if $t < 1$, then increases in n reduce g for low v but increase g for high v. Such a change in strategy might be viewed as low-value players placing greater reliance on high-value players to "step up to the plate" and exert even greater effort than before.

To understand how changes in team size affect efforts, we now focus on the distributions of efforts, given N teams. The cdf of an individual player's effort when n players are on a team is given by

$$H(\gamma; n) \equiv \Pr(g(v) \leq \gamma) = \Pr\left(v \leq \left(\frac{Nn + 1 - t}{(N-1)nt} \gamma \right)^{\frac{1}{Nnt+1-t}} \right) = \left(\frac{Nn + 1 - t}{(N-1)nt} \gamma \right)^{\frac{t}{Nn+1-t}}.$$

Therefore, the cdf of a team's best shot is given by

$$H^{BS}(\gamma\,;\, n) = (H(\gamma; n))^n = \left(\frac{Nnt + 1 - t}{(N-1)nt} \gamma \right)^{\frac{nt}{Nnt+1-t}}. \tag{10}$$

For this power function example, the following proposition extends Theorem 1 and Proposition 2 to show the contest's and teams' best-effort cdfs can be ordered by first-order stochastic dominance (FOSD) as n increases.

[22] By Proposition 1, if $t < 1$ then the only semi-symmetric equilibrium is the one in which all players on a team are active, while if $t \geq 1$, then there exist additional semi-symmetric equilibria in which fewer (but equally many) players per team are active.

Proposition 3 (Team performance and team size)**.**

1. *Holding fixed the number of players per team, if* $t \leq 1$ *then increasing the number of teams shifts the cdf of the contest best effort rightward. For any* $t > 0$*, as* $N \to \infty$ *the limiting distribution of the contest best effort is the uniform distribution on* $[0, 1]$*.*

2. *Holding fixed the number of teams* $N \geq 2$*, the effect of increasing team size is as follows:*

   (a) *If* $t = 1$*, then* $H^{BS}(\gamma) = \left(\frac{N}{N-1}\gamma\right)^{1/N}$ *on* $[0, (N-1)/N]$*, independent of* n*.*

   (b) *If* $t < 1$*, then increasing* n *shifts* H^{BS} *rightward, with limit* $\left(\frac{N}{N-1}\gamma\right)^{1/N}$ *on* $[0, (N-1)/N]$*.*

   (c) *If* $t > 1$*, then increasing* n *shifts* H^{BS} *leftward, with limit* $\left(\frac{N}{N-1}\gamma\right)^{1/N}$ *on* $[0, (N-1)/N]$*.*

Theorem 1 already showed that when teams have at least two players, increasing the number of competing teams improves the expected performance of the contest at the symmetric equilibrium. For concave F, Proposition 3 strengthens this result to improvement by FOSD; however, for convex F ($t > 1$), the cdfs of the contest best shot are not generally ordered by FOSD.

Because the cdf of the contest best effort is simply $(H^{BS})^N$, Proposition 3 implies that, as team size grows, the cdf of the contest best effort shifts leftward or rightward according to whether $t \gtrless 1$, converging to the uniform distribution on $[0, (N-1)/N]$ as $n \to \infty$. It is worth noting that as $n \to \infty$, the maximum value on *all* teams is essentially 1. Nevertheless, because the incentive to free ride remains, the range of bids is bounded away from 1; therefore, players receive positive payoffs.

We next investigate the individual efforts underlying a team's best shot. Theorem 1 implies that if F is convex, then increasing n reduces an individual efforts in the sense of FOSD. However, when F is strictly concave, as n increases, a player's effort cdfs cannot be ordered by stochastic dominance. So we turn to expected efforts.

For the remainder of this subsection, we assume there are only two teams, that is, $N = 2$*.* An individual's expected effort, $\mathrm{E}[\, g \,|\, n \,]$, can be directly obtained from the formula (9). Total effort is a common measure of performance in contests. The next result shows that convexity of F is far from necessary for individual expected efforts to be decreasing with n. Also, convexity is not sufficient for a teams' total expected effort to be decreasing with n. Rather, the free-riding effect of convexity must offset the effort contributed by the additional player, and this requires the convexity to be sufficiently pronounced.

Proposition 4. *Suppose* $N = 2$*.*

1. *If* $t > 0.296535$*, then* $\mathrm{E}[\, g \,|\, n \,]$ *is strictly decreasing in* n*. If* $t < 0.296535$*, then* $\mathrm{E}[\, g \,|\, n \,]$ *initially increases with* n *and then strictly decreases.*

2. *If $t \geq 2$, then $\mathrm{E}[\,ng\,|\,n\,]$ strictly decreases with n. If $t < 1.693$, then $\mathrm{E}[\,ng\,|\,n\,]$ strictly increases with n. If $1.693 < t < 2$, then $\mathrm{E}[\,ng\,|\,n\,]$ initially decreases with n and then strictly increases. In all cases, $\lim_{n\to\infty} \mathrm{E}[\,ng\,|\,n\,] = 0.25$.*

Note that for $t \in (1, 1.693)$ total expected team effort increases with n, but, nevertheless, team and contest performance decrease, reflecting the disconnect between the *sum* of efforts and the *maximum* of efforts.

The following proposition summarizes the effect of team size on expected utility.

Proposition 5. *Suppose $N = 2$.*

1. *If $t > 0.640388$, then $\mathrm{E}[\,U^*(v)\,|\,n\,]$ is strictly increasing in n.*

2. *If $t < 0.640388$, then $\mathrm{E}[\,U^*(v)\,|\,n\,]$ initially decreases with n and then strictly increases.*

In Figure 1 the thin line depicts the locus where $\partial \mathrm{E}[\,g\,|\,n\,]/\partial n = 0$; to the right of this locus this derivative is negative, and to the left positive. The negative correlation effect of an additional player ensures that along this thin line, it must be that $\partial \mathrm{E}[\,U^*(v)\,|\,n\,]/\partial n < 0$. In order for the addition of a player to each team not to reduce expected payoffs, it must be that individual expected effort falls, implying the thick line, where the effect on expected utility is null, will lie to the right of the thin line. Right of this thick (red) curve, where $\partial \mathrm{E}[\,U^*(v)\,|\,n\,]/\partial n = 0$, this derivative is positive and to the left it is negative. Thus, for example, if $t \geq 1$, then part 2(d) of Theorem 1 shows expected utility increases with n; the figure, shows this is true for a larger range of t as well. One implication of Proposition 5 is that for "most" contests ($t > 0.640388$) described in this subsection, active players benefit from increasing the common team size. And even for lower values of t, increasing the size of teams increases individual payoffs if there are sufficiently many active players. In the region between the two curves, increasing team size reduces individuals' expected efforts *and* their expected payoffs—here the reduction in the beneficial correlation outweighs the cost savings, leading to lower expected payoffs as team size increases.

4 Asymmetric contests between two groups

4.1 A general analysis

Now suppose $N = 2$ and group 1 has m members with marginal cost c_1, group 2 has n members and cost c_2, and all values remain independently and identically distributed according to F. We derive the semi-symmetric equilibrium in which all players are active. Let g_i denote the common strategy of team i members. Given the findings of the previous section, we look for an equilibrium in which strategies are continuous, start from

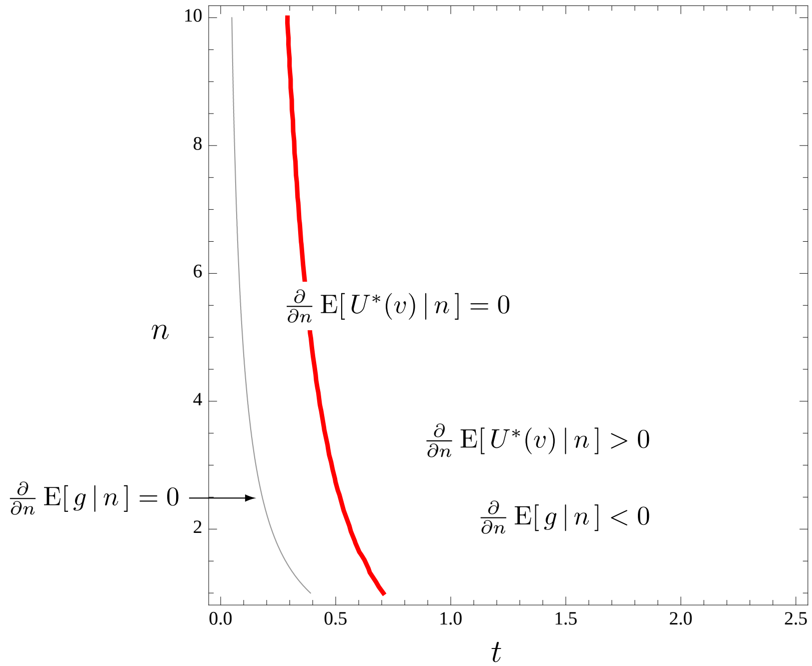

Figure 1: Effects of increasing n on expected individual expected utility, $N = 2$.

zero, and are strictly increasing once they become positive. Let φ_i denote the inverse of g_i, $i = 1, 2$. We begin by considering a member of team 1. Let F^M denote the cdf of the maximum of $m-1$ draws from F and let P^M denote the cdf of the maximum of n draws from F; let f^M and p^M denote the associated densities. Then a player on team 1 with value v acting like a type v' player has payoff

$$U_1(v', v) = -c_1 g_1(v') + v\left[F^M(v')P^M(\varphi_2(g_1(v'))) + \int_{v'}^{\bar{v}} P^M(\varphi_2(g_1(y)))\, f^M(y)\, dy\right].$$

We have

$$\begin{aligned}\frac{\partial U_1(v', v)}{\partial v'} &= -c_1 g_1'(v') + v\left[F^M(v')p^M(\varphi_2(g_1(v')))\varphi_2'(g_1(v'))g_1'(v')\right] \\ &= g_1'(v')\left[-c_1 + v\big(F(v')\big)^{m-1} n\big(F(\varphi_2(g_1(v')))\big)^{n-1} f(\varphi_2(g_1(v')))\varphi_2'(g_1(v'))\right];\end{aligned}$$

if we let $\gamma = g_1(v)$ and recognize that $v = \varphi_1(\gamma)$ where g_1 is strictly increasing, then the equilibrium condition, evaluated at $v' = v$, becomes

$$c_1 = n\varphi_1(\gamma)\big(F(\varphi_1(\gamma))\big)^{m-1}\big(F(\varphi_2(\gamma))\big)^{n-1} f(\varphi_2(\gamma))\,\varphi_2'(\gamma); \tag{11}$$

a symmetric argument for a team-2 player shows

$$c_2 = m\varphi_2(\gamma)\big(F(\varphi_1(\gamma))\big)^{m-1}\big(F(\varphi_2(\gamma))\big)^{n-1} f(\varphi_1(\gamma))\,\varphi_1'(\gamma). \tag{12}$$

An appropriate solution to the system (11)–(12) will characterize a semi-symmetric equilibrium to this two-team contest. The following proposition assures existence and uniqueness of such a solution.

Proposition 6. *Suppose the pdf f is strictly positive on $(\underline{v}, \bar{v}]$ and suppose f is continuously differentiable with bounded derivative on $(\underline{v}, \bar{v}]$. Suppose $c_1 m \le c_2 n$. Then there exists a unique solution (φ_1, φ_2) to the system* (11)–(12) *such that $\varphi_1(0) = \underline{v}$ and, for some $\bar{g} > 0$, $\varphi_1(\bar{g}) = \varphi_2(\bar{g})$.*

From (11) and (12) we clearly see that marginal costs and the numbers of players per team play a key role in equilibrium, as highlighted in the next proposition.

Proposition 7 (Players on the more efficient and smaller team work harder)**.** *Suppose the pdf f is strictly positive on $(\underline{v}, \bar{v}]$ and suppose f is continuously differentiable with bounded derivative on $(\underline{v}, \bar{v}]$. If $c_1 m < c_2 n$, then $g_1(v) > g_2(v)$ for all $v \in (\underline{v}, \bar{v})$.*

First, the effort ranking in Proposition 7 is intuitive when teams are of equal size: individuals on the team with lower cost of effort use a more aggressive strategy. Second, when $c_1 = c_2$, Proposition 7 suggests a real tradeoff. Individuals on the smaller team exert greater effort, for a given v, which tends to favor the smaller team. However, team performance is determined by the highest-value order statistic, which favors the larger team; that is, even though a team-2 member uses a lower strategy function, team 2's highest player value is likely to exceed that of team 1. Thus, it is not clear whether the larger or smaller team is favored in this contest of equal players but unequal teams. We explore these issues in the following example, showing in part that either the smaller or the larger group may perform better in the sense of first-order stochastic dominance, so either team may be more likely to win the contest.

4.2 A family of examples: revisiting the power function distribution

Throughout this subsection, we assume $c_1 = c_2 = 1$ and players' values are distributed according to $F(v) = v^t$ on $[0, 1]$. Given equal costs and this cdf, we equate (11) and (12) and then rearrange terms to obtain

$$n\,(\varphi_2(\gamma))^{t-2}\,\varphi_2'(\gamma) = m\,(\varphi_1(\gamma))^{t-2}\,\varphi_1'(\gamma). \tag{13}$$

We consider (13) in turn for $t = 1$ and $t \neq 1$.

Case 1: $F(v) = v$. Condition (13) can be integrated to obtain $n \log(\varphi_2) = A + m \log(\varphi_1)$ for some constant A. Let $\bar{g}$ denote the (common) maximum effort under each strategy; then $\varphi_i(\bar{g}) = 1$, so $\log(\,\varphi_i(\bar{g})\,) = 0$, implying $A = 0$. Thus, we have

$$(\varphi_2(\gamma))^n = (\varphi_1(\gamma))^m. \tag{14}$$

Note that $\Pr(\text{group 1's best shot} \leq \gamma) = \Pr(\max v \leq \varphi_1(\gamma)) = (\varphi_1(\gamma))^m$, and a similar result holds for group 2. Thus, (14) shows the cdfs of the teams' best shots are the same. Consequently, each team has an equal chance of winning, even though they differ in size.

To identify the equilibrium strategies, we use (11) and (14) to obtain

$$1 = n\varphi_1\big(\varphi_1\big)^{m-1}\big(\varphi_2\big)^{n-1}\varphi_2' = n\big(\varphi_2\big)^n\big(\varphi_2\big)^{n-1}\varphi_2' = \frac{1}{2}\frac{d}{d\gamma}\big(\varphi_2\big)^{2n}. \tag{15}$$

Either g_1 or g_2 must be strictly increasing from 0, for otherwise a player with value near 0 could effect a discrete increase in the probability of winning at negligible cost. Therefore, either $\varphi_1(0) = 0$ or $\varphi_2(0) = 0$; by (14) it follows that both equal 0. Integrating the extremes of (15) and using the boundary condition $\varphi_2(0) = 0$ we obtain $\gamma = \frac{1}{2}\big(\varphi_2\big)^{2n}$. Now, recalling $\gamma = g_2(v)$ and $\varphi_2(\gamma) = v$, we see

$$g_2(v) = \frac{1}{2}v^{2n}; \tag{16}$$

similarly,

$$g_1(v) = \frac{1}{2}v^{2m}. \tag{17}$$

Case 2. $F(v) = v^t$, $t \neq 1$. Integrating (13) and using the boundary condition $\varphi_i(\bar{g}) = 1$, we obtain

$$n\varphi_2^{t-1} = (n-m) + m\varphi_1^{t-1}. \tag{18}$$

As above, either g_1 or g_2 must be strictly increasing from $v = 0$.

If $t > 1$, then from (18) it must be $\varphi_1(0) = 0$ (i.e., $g_1(0) = 0$), and, therefore,

$$\varphi_2(0) = \left(\frac{n-m}{n}\right)^{\frac{1}{t-1}} = \left(1 - \frac{m}{n}\right)^{\frac{1}{t-1}}. \tag{19}$$

Thus, for $t > 1$, the larger team places an atom on 0 effort, and the size of this atom increases with the relative disparity in team size.

	$F(v)=\sqrt{v}$	$F(v)=v$	$F(v)=v^2$
$m=n=1$	$g_1=g_2=\frac{1}{3}v^{3/2}$	$g_1=g_2=\frac{1}{2}v^2$	$g_1=g_2=\frac{2}{3}v^3$
$m=1$	$g_1=4\left(2\sqrt{v}+\frac{1-6(1+\sqrt{v})}{(1+\sqrt{v})^2}+5-6\log(1+\sqrt{v})\right)$	$g_1=\frac{1}{2}v^2$	$g_1=\frac{v^2}{4}\left(\frac{1}{2}+v+\frac{3}{4}v^2+\frac{1}{5}v^3\right)$
$n=2$	$g_2=24\log\left(1-\frac{\sqrt{v}}{2}\right)+\frac{16}{2-\sqrt{v}}+8\sqrt{v}+v-8$	$g_2=\frac{1}{2}v^4$	$g_2=\frac{1}{80}+\frac{8}{5}v^5-v^4$ on $[\frac{1}{2},1]$
$m=n=2$	$g_1=g_2=\frac{2}{5}v^{5/2}$	$g_1=g_2=\frac{1}{2}v^4$	$g_1=g_2=\frac{4}{7}v^7$

Table 1: Equilibrium strategies when v has a power function distribution on $[0,1]$; m players are active in group 1 and n are active in group 2.

If $t<1$, then we multiply (18) by $(\varphi_1\varphi_2)^{1-t}$ to obtain

$$n\varphi_1^{1-t}=(n-m)\varphi_1^{1-t}\varphi_2^{1-t}+m\varphi_2^{1-t}. \tag{20}$$

Next we show that *both* g_1 and g_2 strictly increase from 0: taking limits in (20), we see

$$\lim_{\gamma\downarrow 0}\ n\varphi_1^{1-t}=\lim_{\gamma\downarrow 0}\ (n-m)\varphi_1^{1-t}\varphi_2^{1-t}+m\varphi_2^{1-t}=\lim_{\gamma\downarrow 0}\ m\varphi_2^{1-t}, \tag{21}$$

where the second equality follows because either $g_1(0)=0$ or $g_2(0)=0$, i.e., either $\varphi_1(0)=0$ or $\varphi_2(0)=0$. Since either the left extreme or right extreme of (21) equals 0, both must equal 0. That is, when $t<1$ both groups' equilibrium strategies will be strictly increasing from 0.

The two parts of Case 2 again illustrate nicely the importance of F as convex or concave. When F is convex—which promotes free riding—some types of players on the larger team choose to be inactive; but when F is concave, all types of all players are active.

Table 1 reports equilibrium strategies for symmetric and asymmetric cases for convex and concave cdf examples.[23] These examples make clear that, when teams differ in size, derivation of explicit equilibrium strategies is complex at best.

As a final step in our analysis, we compare the distributions of the two groups' best shots and examine which team is more likely to win. Proposition 8 shows how relative team performance depends on relative team size and the nature of private information.

Proposition 8 (Team performance and the role of relative team size)**.** *Suppose group 1 has m members and group 2 has n, with $m<n$. Let the distribution of each player's value be given by $F(v)=v^t$ on $[0,1]$, where $t>0$. Let H_i^{BS} denote the cdf of group i's best-shot when players follow the equilibrium strategies derived*

[23] The first and third rows are calculated using (9). The asymmetric uniform-distribution case strategies has equilibrium strategies given by (16) and (17). Strategies in the remaining two cells of the table are derived in the Appendix.

above, $i = 1, 2$.

1. *If* $t = 1$ *then* $H_1^{BS}(\gamma) = H_2^{BS}(\gamma)$ *for all* γ.

2. *If* $t > 1$ *then* $H_1^{BS}(\gamma) < H_2^{BS}(\gamma)$ *for all* γ *such that* $H_1^{BS}(\gamma) < 1$.

3. *If* $t < 1$ *then* $H_1^{BS}(\gamma) > H_2^{BS}(\gamma)$ *for all* γ *such that* $0 < H_1^{BS}(\gamma) < 1$.

From Proposition 8 it follows that either the smaller or the larger team may be more likely to win. If $t > 1$ (the cdf of individual values is convex), then the order-statistic advantage of the larger group is not so significant relative to the free-riding incentives, and the smaller group is more likely to win.[24] But if $t < 1$ (the cdf of individual values is concave), the order-statistic advantage to the larger group is relatively more important, and the larger group has the advantage. This accords with our earlier intuition, developed after Proposition 2.

To better understand the role of relative group size and the nature of uncertainty, we next calculate team 1's probability of winning. Fortunately, this can be done without explicit closed-form equilibrium strategies. Let $\bar{g}$ denote the maximum effort under the specified equilibrium strategies. Then, for any $\gamma \in (0, \bar{g})$, we calculate the cdf of team 1's best effort as

$$H_1^{BS}(\gamma) = \Pr(g_1(v_k) \le \gamma, k = 1, \ldots, m) = \Pr(v_k \le \varphi_1(\gamma), k = 1, \ldots, m) = (\varphi_1(\gamma))^{mt};$$

similarly, $H_2^{BS}(\gamma) = (\varphi_2(\gamma))^{nt}$. For $t \neq 1$ we have

$$\begin{aligned}
\Pr(\text{team 1 wins}) &= \int_0^{\bar{g}} H_2^{BS}(\gamma)\, dH_1^{BS}(\gamma) \\
&= \int_0^{\bar{g}} \left[\left(1 - \frac{m}{n}\right) + \frac{m}{n}(\varphi_1(\gamma))^{t-1}\right]^{\frac{nt}{t-1}} mt(\varphi_1(\gamma))^{mt-1}\, \varphi_1'(\gamma)\, d\gamma \qquad \text{(using (18))} \\
&= \int_0^1 \left[\left(1 - \frac{m}{n}\right) + \frac{m}{n}y^{t-1}\right]^{\frac{nt}{t-1}} mty^{mt-1}\, dy, \qquad (22)
\end{aligned}$$

where in (22) we have used the change of variable $y = \varphi_1(\gamma)$. For different values of m, n, and t, the integral in (22) can be calculated, and Tables 2 and 3 reveal the patterns due to relative team size. We see that if F is convex ($t = 2$), then increasing n, and thereby increasing the difference in team size, enhances team 1's chance of winning. Moreover, team 1's chance of winning is greatest when this difference is size is largest: $m = 1$ and $n \to \infty$. In contrast, if F is concave ($t = 1/2$), then increasing n reduces team 1's chance of winning, and team 1's chance of winning is least when this difference is size is largest: $m = 1$ and $n \to \infty$.

[24] This complements the finding in the full-information model of Barbieri *et al.* (2013) that, when players have a common value of winning but teams differ in size, the smaller team has the advantage and is more likely to win.

It is worth emphasizing that even in cases favoring the larger team (e.g., $t = 1/2$), the probability that the larger team wins remains bounded away from 0 even as n becomes arbitrarily large.

Table 2: Team 1's probability of winning, given $t = 2$.

$\downarrow m \setminus n \rightarrow$	1	2	3	4	5	10	20	∞
1	.5	.5375	.548256	.553356	.556333	.562111	.564916	.567668
2	-	.5	.511608	.51718	.520448	.526816	.529919	.532967
3	-	-	.5	.505616	.508921	.515382	.518537	.521643
4	-	-	-	.5	.503311	.509795	.512967	.516091
5	-	-	-	-	.5	.506941	.509669	.512802
10	-	-	-	-	-	.5	.503184	.506327
20	-	-	-	-	-	-	.5	.506327

Table 3: Team 1's probability of winning, given $t = 1/2$.

$\downarrow m \setminus n \rightarrow$	1	2	3	4	5	10	20	∞
1	.5	.454823	.438275	.429772	.424609	.414177	.408922	.403653
2	-	.5	.48278	.473764	.468232	.456917	.451147	.445314
3	-	-	.5	.490913	.48531	.473784	.467871	.46187
4	-	-	-	.5	.494383	.482789	.47682	.470748
5	-	-	-	-	.5	.488382	.482388	.476279
10	-	-	-	-	-	.5	.493977	.487818
20	-	-	-	-	-	-	.5	.493829

5 Conclusion

This paper is novel for its introduction of private information into group contests. We have supposed that players have private values for their own teams winning the contest. With the introduction of private information, we have modelled a team's performance as equalling the "best shot" of its members. This assumption offers considerable tractability and may be a reasonable description of some contests, particularly those that involve generating bold, innovative ideas. With such non-convex within-team technology, some full-information models of group contests with constant marginal costs find only one "champion" is active per team.[25] In contrast, we have shown that when information is private we may expect a fuller participation of players—even with constant marginal costs of effort—because no player can be sure of others' efforts.

We targeted two prominent features of group contests: the number of groups competing and the size of each group. Focussing first on the situation where teams behave symmetrically, we showed that for "true" groups (those with at least two members), increasing the number of competing teams is sure to increase

[25]See for example, Chowdhury *et al.* (2013a).

contest performance, though expected team performances may fall. By contrast, the effects of increasing team size are more complex, as two opposing forces are at work. Increasing all teams' size increases the scope for free riding within a team, but it also introduces additional rivals against whom one must compete. The relative strength of these forces depends on the nature of private information. We showed that convexity of the cdf of players' values strengthens the incentives for free riding—in this case, having larger teams leads to lower individual, contest, and team performance. Moreover, these reductions in effort are sufficiently strong that players' expected utilities increase with team size. By contrast, when the cdf of values is concave we found team and contest performance improve with team size. Interestingly, we showed in a family of examples that, for concave cdfs, larger teams can lead to lower individual efforts *and* to lower expected utility. This arises from yet another consequence of larger team size, namely, a reduced correlation between a player's value of winning and his interim probability of winning. Given teams act symmetrically, *ex ante* they all have an equal chance of being the contest winner. This given probability of winning is more beneficial when more strongly correlated with an individual's realized value of winning—and it is precisely this correlation that is reduced as team size increases.

Finally, we considered contests between two groups of different size but with otherwise symmetric players. The question naturally arises whether the larger or smaller group more effectively pursues its goals. We showed that members of the smaller team are sure to use more aggressive effort strategies. Nevertheless, the smaller team is not necessarily favored to win, since the larger team benefits from an order statistic effect—many players acting half-heartedly may still produce a better best shot than a small team where individuals work diligently. But for a concave value cdf free-riding incentives are weaker and the order-statistic advantage of the larger team dominates. For a family of examples, we studied the balance of these effects in greater detail and found that the nature of the value cdf is again critical. Here the results reflect the earlier forces: a convex value cdf promotes greater free riding, and this is precisely the case in which the larger team is less likely to win the contest. Thus, in our examples the smaller (larger) team is favored to win the contest when the value cdf is convex (concave), and this advantage increases with an increasing disparity in team size.

Despite our interest in the overall performance of the contest, we have not analyzed here the possibility that a designer might endogenously choose team size and number of teams simultaneously. Rather, we have taken these variables to be exogenous. Similarly, information acquisition issues are beyond the scope of this paper. A full study of these issues is left for future research. In addition, private information is of interest for group contests that employ the weakest-link aggregator, as it is done in the full-information models of Chowdhury *et al.* (2013b) and Chowdhury and Topolyan (2013). Exploring the consequences of private information in this framework is the subject of current research.

Appendix

Proof of Proposition 1. (i) We begin by verifying that when all players are active, the strategy g in (5) constitutes an equilibrium. Consider one player, supposing all others use this g. Then, using (4) in (3), for this player the partial derivative (3) becomes

$$\frac{\partial U(v', v)}{\partial v'} = \frac{n(N-1)}{nN-1} f^M(v')(v - v'), \tag{23}$$

which is strictly positive for $v' < v$ and strictly negative for $v' > v$. Therefore, the unique best response to all other players using $g(\cdot)$ is also to follow $g(\cdot)$.

Consider the possibility that each team has n players but in equilibrium only m are active, where $m \in \{1, \ldots, n-1\}$. If there is a semi-symmetric equilibrium with m active players per team then all active players use the strategy in (5), with m substituting for n, as in (6). However, such strategies form an equilibrium if and only if the inactive players never wish to become active. This is assured if an inactive player with value $\bar{v}$ does not want to exert positive effort.

Now suppose each team has $m \in \{1, \ldots, n-1\}$ active players using the strategy in (6). An inactive player with value $\bar{v}$ need only consider deviation efforts in the interval $(0, g(\bar{v})]$. Consequently, this player with value $\bar{v}$ exerting effort $g(v')$ earns payoff

$$\begin{aligned} U(v', v) &= -g(v') + \bar{v}\left[(F(v'))^{Nm} + \frac{1}{N}\left(1 - (F(v'))^{Nm}\right)\right] \\ &= -g(v') + \bar{v}\left[\frac{1}{N} + \frac{N-1}{N}(F(v'))^{Nm}\right]. \end{aligned}$$

Now, for $g(v') \in (0, g(\bar{v}))$, we have

$$\begin{aligned} \frac{\partial U(v', v)}{\partial v'} &= -g'(v') + \bar{v}(N-1)m(F(v'))^{Nm-1}f(v') \\ &= -\frac{m(N-1)}{mN-1} v' f^M(v') + \bar{v}(N-1)m(F(v'))^{Nm-1}f(v') && \text{(by (4))} \\ &= -m(N-1)v'(F(v'))^{Nm-2}f(v') + \bar{v}(N-1)m(F(v'))^{Nm-1}f(v') && (24) \\ &= m(N-1)(F(v'))^{Nm-2}f(v')\bar{v}\left[F(v') - \frac{v'}{\bar{v}}\right], && (25) \end{aligned}$$

where (24) uses the fact that here $f^M(v') = (Nm-1)(F(v'))^{Nm-2}f(v')$.

(ii) Under the assumption that there exists $\varepsilon > 0$ such that $F(v) \geq v/\bar{v}$ for all $v \in (\underline{v}, \underline{v} + \varepsilon)$, with strict inequality holding for some $v' \in (\underline{v}, \underline{v} + \varepsilon)$, it follows that in (25) the term in square brackets is nonnegative for all $v \in (\underline{v}, \underline{v}+\varepsilon)$ and is strictly positive over a nondegenerate interval in $(\underline{v}, \underline{v}+\varepsilon)$. Consequently, the inactive player with value $\bar{v}$ would strictly prefer exerting effort $g(\underline{v}+\varepsilon)$ to remaining inactive. By continuity a positive mass of types near $\bar{v}$ would also choose not to remain inactive, implying the proposed strategy together with $n-m$ inactive players does not constitute a semi-symmetric equilibrium. This argument is valid for all $m \in \{1, \ldots, n-1\}$, so there is no semi-symmetric equilibrium with inactive players.

(iii) If instead F first-order stochastically dominates the uniform distribution on $[0, \bar{v}]$, then in (25) the term in square brackets is nonpositive, and for some v' it is strictly negative. It follows that $U(v')$ is nonincreasing over $[\underline{v}, \bar{v}]$, implying an inactive player can do no better than to remain inactive, even when his value is $\bar{v}$. □

Proof of Theorem 1. Proof of part 1(a). Here we use an alternative calculation of interim utility. Equations (2) and (3) together with the Envelope Theorem imply we can rewrite the equilibrium interim utility above as

$$U^*(v; n, N) = U^*(\underline{v}; n, N) + \int_{\underline{v}}^{v} p(z; n, N)\, dz, \tag{26}$$

where $U^*(\underline{v}; n, N) = \underline{v} \cdot (n-1)/(nN-1)$. Next we show that $p(x; n, N)$ is also a strictly decreasing function of N for all $x \in (\underline{v}, \bar{v})$. These two properties show through (26) that U^* is strictly decreasing in N.

Observe that

$$\frac{\partial p(v; n, N)}{\partial N} = \frac{n}{(nN-1)^2}\Big[-(n-1) + (F(v))^{nN-1}\Big(n-1+n(N-1)\log\big((F(v))^{nN-1}\big)\Big)\Big]. \tag{27}$$

Because $(F(v))^{nN-1}$ takes on the range of values $(0,1)$ as v varies over $(\underline{v}, \bar{v})$, to show $\partial p/\partial N < 0$, it suffices to show that

$$a(z) \equiv -(n-1) + z[n-1+n(N-1)\log(z)] < 0$$

for all $z \in (0,1)$. Next observe that $a(0) = -(n-1) < 0$, $a(1) = 0$, and $a''(z) = n(N-1)/z > 0$ for all $z > 0$. This convexity of a implies $a(z) < 0$ for all $z \in [0,1)$, as was to be shown.

Proof of part 1(b). Fix $n \geq 2$.

Step 1: Rewrite the quantity of interest using the change of variable $y = F(x)$ and a density $\phi(y;N)$. The expected best effort in the contest equals

$$\begin{aligned}\int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^{nN}\, dv &= \int_{\underline{v}}^{\bar{v}} g(v) nN (F(v))^{nN-1} f(v)\, dv \\ &= \int_{\underline{v}}^{\bar{v}} \left(n(N-1)\int_{\underline{v}}^{v} x(F(x))^{Nn-2} f(x)\, dx\right) nN(F(v))^{nN-1} f(v)\, dv\end{aligned}$$

$$
\begin{aligned}
&= \int_{\underline{v}}^{\bar{v}} n^2 N(N-1) \left(\int_{x}^{\bar{v}} (F(v))^{nN-1} f(v)\, dv \right) x (F(x))^{Nn-2} f(x)\, dx \\
&= \int_{\underline{v}}^{\bar{v}} n(N-1) \left(1 - (F(x))^{nN}\right) x (F(x))^{Nn-2} f(x)\ dx \\
&= \int_0^1 F^{-1}(y)\, \phi(y;N)\, dy, \qquad (28)
\end{aligned}
$$

where the second equality uses the equilibrium strategy in (5), the third follows from interchanging the order of integration, and the last from the change of variables $y = F(x)$ and the definition $\phi(y;N) \equiv n(N-1)(1-y^{nN})y^{Nn-2}$.

The method of proof will be to use first-order stochastic dominance (FOSD), regarding $F^{-1}(y)$ as a "utility function" and $\phi(y;N)$ as a density.

Step 2: Verify ϕ*(y;N)* is indeed a density and define its cdf. Integration of ϕ over $[0,1]$ yields

$$
\int_0^1 \phi(y;N)\, dy = \frac{N(N-1)n^2}{(Nn-1)(2Nn-1)} < 1,
$$

leading us to define the cdf H on $[0,1]$ by

$$
H(y;N) = \left(1 - \frac{N(N-1)n^2}{(Nn-1)(2Nn-1)}\right) + \int_0^y n(N-1)(1-x^{Nn})x^{Nn-2}\, dx.
$$

Step 3: Show that H(· ; N+1) FOSD H(· ; N). We show $H(y;N+1) < H(y;N)\ \forall y \in [0,1)$. First we find that

$$
\frac{\partial H(0;N)}{\partial N} = -\frac{n^2\Big((2N-1) + n(2n-3)N^2\Big)}{(nN-1)^2(2nN-1)^2} < 0
$$

for all $n \geq 2$. Therefore, $H(0;N+1) - H(0;N) < 0$; and, of course, $H(1;N+1) - H(1;N) = 0$. Furthermore,

$$
\frac{d}{dy}\left(H(y;N+1) - H(y;N)\right) = -\left(\frac{1-(y^n)^N}{1-(y^n)^{N+1}} - \frac{N}{N-1}y^n\right) n(N-1)(1-(y^n)^{N+1})y^{Nn-2},
$$

and because $(1-(y^n)^N)/(1-(y^n)^{N+1})$ is a strictly decreasing function of y^n for $y \in [0,1]$,[26] this derivative establishes that $H(y;N+1) - H(y;N)$ is a strictly quasi-convex function of y. Therefore, $H(y;N+1) - H(y;N) < 0$ for any $y \in (0,1)$, thus establishing the desired FOSD ordering.

Step 4: Conclude the proof using Step 3. We have

$$
\mathrm{E}[\, g(v) \,|\, N+1\,] + H(0;N+1)\underline{v} = nN \int_0^1 (1 - ny^{N+1})y^{(N+1)n-2} F^{-1}(y)\, dy + H(0;N+1)\underline{v}
$$

[26] To see this function is decreasing, we note $((1-z^N)/(1-z^{N+1}))' = z^{N-1}(1-z^{N+1})^{-2}(-N + (N+1)z - z^{N+1})$. Define $a(z) \equiv -N + (N+1)z - z^{N+1}$. Because $a(1) = 0$ and $a'(z) = (N+1)(1-z^N) > 0$ for all $z < 1$, it follows that $a(z) < 0$ for all $z \in [0,1)$.

$$
\begin{aligned}
&= \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; N+1)\,\big] \\
&\geq \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; N)\,\big] \\
&= n(N-1)\int_0^1 (1-ny)y^{Nn-2}F^{-1}(y)\,dy + H(0;N)\underline{v} \\
&= \mathrm{E}[\, g(v) \,|\, N\,] + H(0;N)\underline{v},
\end{aligned}
$$

where the inequality follows from FOSD (Step 3) since F^{-1} is increasing. Rearranging extremes of the previous system, we have

$$
\begin{aligned}
\mathrm{E}[\, g(v) \,|\, N+1\,] &\geq \mathrm{E}[\, g(v) \,|\, N\,] + \big(H(0;N) - H(0;N+1)\big)\underline{v} \\
&\geq \mathrm{E}[\, g(v) \,|\, N\,], \qquad (29)
\end{aligned}
$$

where the second inequality follows from $H(0;N) > H(0;N+1)$ when $n \geq 2$. Now (29) establishes the desired result that the overall expected best effort increases with the number of teams, holding constant the number of members per team.

Proof of part 1(c). The proofs of parts 1(c), 2(a), 2(b), and 2(c) follow the same sequence of six steps.

Step 1: Rewrite the quantity of interest using the change of variable $y = F(x)$ and a density $\phi(y;N)$. A team's expected best effort is equal to

$$
\begin{aligned}
\int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^n\,dv &= \int_{\underline{v}}^{\bar{v}} g(v)n(F(v))^{n-1}f(v)\,dv \\
&= \int_{\underline{v}}^{\bar{v}} \left(n(N-1)\int_{\underline{v}}^{v} x(F(x))^{Nn-2}f(x)\,dx\right) n(F(v))^{n-1}f(v)\,dv \\
&= \int_{\underline{v}}^{\bar{v}} n^2(N-1)\left(\int_{x}^{\bar{v}} (F(v))^{n-1}f(v)\,dv\right) x(F(x))^{Nn-2}f(x)\,dx \\
&= \int_{\underline{v}}^{\bar{v}} n(N-1)\big(1-(F(x))^n\big)x(F(x))^{Nn-2}f(x)\;dx \\
&= \int_0^1 F^{-1}(y)\,\phi(y;N)\,dy, \qquad (30)
\end{aligned}
$$

where the second equality uses the strategy in (5), the third follows from interchanging the order of integration, and the last uses the change of variable and the definition $\phi(y;N) = n(N-1)(1-y^n)y^{Nn-2}$.

Step 2: Verify $\phi(y;N)$ is indeed a density and define its cdf. Integration of ϕ over $[0,1]$ yields

$$
\int_0^1 \phi(y;N)\,dy = \frac{(N-1)n^2}{(Nn-1)(nN+n-1)} < 1,
$$

leading us to define the cdf H on $[0,1]$ by

$$H(y;N)=\left(1-\frac{(N-1)n^2}{(Nn-1)(nN+n-1)}\right)+\int_0^y n(N-1)(1-x^n)x^{Nn-2}\,dx. \tag{31}$$

Step 3: Show the average *y* under *H(y; N)* (weakly) decreases in *N*. We have

$$\int_0^1 y\,dH(y;N)=\frac{N-1}{N(N+1)}, \tag{32}$$

which is decreasing in N.

Step 4: Establish *H(y; N+1)–H(y; N)* starts strictly positive, ends at zero, and has only one sign change for *y*$\in[\mathbf{0},\mathbf{1}]$. Observe that

$$H(0;N+1)-H(0;N)=\frac{((N-2)n+1)n^2}{(nN-1)(nN+n-1)(nN+2n-1)}>0;$$

so $H(0;N+1)-H(0;N)>0$. Next we have

$$\begin{aligned}\frac{d}{dy}\Big(H(y\,;\,N+1)-H(y\,;\,N)\Big)&=-n(1-y^n)y^{nN-2}(N-1-Ny^n)\\ &\begin{cases}<0 & \text{for } y\in\left(0,\left(\frac{N-1}{N}\right)^{1/n}\right)\\ >0 & \text{for } y\in\left(\left(\frac{N-1}{N}\right)^{1/n},1\right).\end{cases}\end{aligned} \tag{33}$$

Because $H(0\,;\,N+1)-H(0\,;\,N)>0$ and, of course, $H(1\,;\,N+1)-H(1\,;\,N)=0$, (33) implies that there is unique $y^c(N)\in(0,1)$ such that, for any $y\in(0,1)$,

$$H(y\,;\,N+1)-H(y\,;\,N)\gtrless 0 \iff y\lessgtr y^c(N). \tag{34}$$

Step 5: Show that *H*$(\cdot$;*N)* second-order stochastically dominates *H*$(\cdot$;*N+1)*. We show

$$\int_0^y H(x;N+1)\,dx-\int_0^y H(x;N)\,dx>0 \qquad \forall y\in(0,1). \tag{35}$$

The validity of (35) follows immediately from (34) for any $y\in(0,y^c(N)]$. So now consider $y\in(y^c(N),1]$. Because the difference of integrals in (35) is strictly decreasing over this interval (by (34)), we see for any

$y \in (y^c(N), 1]$ that

$$\begin{aligned}
\int_0^y H(x; N+1)\,dx - \int_0^y H(x; N)\,dx &\geq \int_0^1 H(x; N+1)\,dx - \int_0^1 H(x; N)\,dx \\
&= \int_0^1 (1 - H(x; N))\,dx - \int_0^1 (1 - H(x; N+1))\,dx \\
&= \mathrm{E}[\, y \mid N\,] - \mathrm{E}[\, y \mid N+1\,] \qquad (36) \\
&\geq 0,
\end{aligned}$$

where (36) follows from integrating $\int_0^1 y\, dH(y; \cdot)$ by parts and the final inequality follows from Step 3.

Step 6: Conclude the proof by using SOSD when F^{-1} is assumed concave (i.e., F is assumed convex). Given that F^{-1} is concave (i.e., F is convex), we have

$$\begin{aligned}
\mathrm{E}[\, g(v) \mid N\,] + H(0; N)\underline{v} &= \mathrm{E}\big[\, F^{-1}(y) \mid H(\,\cdot\,; N)\,\big] \\
&\geq \mathrm{E}\big[\, F^{-1}(y) \mid H(\,\cdot\,; N+1)\,\big] \qquad (37) \\
&= \mathrm{E}[\, g(v) \mid N+1\,] + H(0; N+1)\underline{v},
\end{aligned}$$

where (37) follows because $H(\,\cdot\,; N)$ SOSD $H(\,\cdot\,; N+1)$. Rearranging extremes of the previous system, we have

$$\begin{aligned}
\mathrm{E}[\, g(v) \mid N\,] &\geq \mathrm{E}[\, g(v) \mid N+1\,] + \big(H(0; N+1) - H(0; N)\big)\underline{v} \\
&\geq \mathrm{E}[\, g(v) \mid N+1\,], \qquad (38)
\end{aligned}$$

where the second inequality follows from $H(0; N+1) > H(0; N)$. Now (38) establishes the desired result that a team's expected best effort decreases with the number of team members.

Proof of part 2(a). **Step 1: Rewrite the quantity of interest using the change of variable $y = F(x)$ and a density $\phi(y;n)$.** We rewrite an individual's effort, using (5), as

$$g(v) = n(N-1) \int_{\underline{v}}^{v} x (F(x))^{Nn-2} f(x)\, dx = \int_0^{F(v)} F^{-1}(y) \phi(y; n)\, dy,$$

where $\phi(y; n) = n(N-1) y^{nN-2}$.

The rest of the proof uses second-order stochastic dominance (SOSD), regarding $F^{-1}(y)$ as a "utility function" and $\phi(y; n)$ as a density.

Step 2: Verify ϕ*(y;n)* is indeed a density and define its cdf. Integration of ϕ over $[0, F(v)]$ yields

$$\int_0^{F(v)} \phi(y;n)\,dy = \frac{n(N-1)(F(v))^{nN-1}}{Nn-1} < 1,$$

leading us to define the cdf H on $[0, F(v)]$ by

$$H(y;n) = \left(1 - \frac{n(N-1)(F(v))^{nN-1}}{Nn-1}\right) + n(N-1)\int_0^y x^{Nn-2}\,dx. \tag{39}$$

Step 3: Show the average *y* under *H(y; n)* decreases in *n*. Integration yields

$$\mathrm{E}[\,y\,|\,n\,] \equiv \int_0^{F(v)} y\,dH(y;n) = \frac{N-1}{N}(F(v))^{nN}, \tag{40}$$

which strictly decreases in n for any $v \in (\underline{v}, \bar{v})$.

Step 4: Establish *H(y; n+1)–H(y; n)* starts strictly positive, ends at zero, and has only one sign change for *y*$\in [\mathbf{0}, \mathbf{1}]$. First, we have

$$\frac{\partial H(0;n)}{\partial n} = \frac{(F(v))^{nN-1}(N-1)\Big[1 - nN(nN-1)\log(\,F(v)\,)\Big]}{(nN-1)^2} > 0,$$

so $H(0;n+1) - H(0;n) > 0$. Second, observe that

$$\frac{d}{dy}\Big(H(y\,;\,n+1) - H(y\,;\,n)\Big) = (N-1)y^{Nn-2}\Big((n+1)y^N - n\Big)$$
$$\begin{cases} < 0 & \text{for } y \in \left(0, \left(\dfrac{n}{n+1}\right)^{1/N}\right) \\ > 0 & \text{for } y \in \left(\left(\dfrac{n}{n+1}\right)^{1/N}, 1\right). \end{cases} \tag{41}$$

Because $H(0;n+1) - H(0;n) > 0$ and $H(F(v);n+1) - H(F(v);n) = 0$, (41) implies that there is unique $y^c(n) \in (0,1)$ such that, for any $y \in (0, F(v))$,

$$H(y\,;\,n+1) - H(y\,;\,n) \gtreqless 0 \iff y \lesseqgtr y^c(n). \tag{42}$$

Step 5: Show that *H*$(\,\cdot\,;$*n) second-order stochastically dominates H*$(\,\cdot\,;$*n+1).* We show

$$\int_0^y H(x;n+1)\,dx - \int_0^y H(x;n)\,dx > 0 \qquad \forall y \in (0, F(v)). \tag{43}$$

The validity of (43) follows immediately from (42) for any $y \in (0, y^c(n)]$. So now consider $y \in (y^c(n), F(v)]$ (this interval is empty for for low v). Because the difference of integrals in (43) is strictly decreasing over this interval (by (42)), we see that, for any $y \in (y^c(n), F(v)]$,

$$
\begin{aligned}
\int_0^y H(x; n+1)\, dx - \int_0^y H(x; n)\, dx &\geq \int_0^{F(v)} H(x; n+1)\, dx - \int_0^{F(v)} H(x; n)\, dx \\
&= \int_0^{F(v)} (1 - H(x; n))\, dx - \int_0^{F(v)} (1 - H(x; n+1))\, dx \\
&= \mathrm{E}[\, y \,|\, n\,] - \mathrm{E}[\, y \,|\, n+1\,] \qquad (44) \\
&> 0,
\end{aligned}
$$

where the final inequality follows from Step 3.

Step 6: Conclude the proof by using SOSD when F^{-1} is assumed concave (i.e., F is assumed convex). We have

$$
\begin{aligned}
g(v; n) + H(0; n)\underline{v} &= n(N-1) \int_0^{F(v)} F^{-1}(y) y^{Nn-2}\, dy + H(0; n)\underline{v} \\
&= \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; n)\,\big] \\
&\geq \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; n+1)\,\big] \qquad (45) \\
&= (n+1)(N-1) \int_0^{F(v)} F^{-1}(y) y^{N(n+1)-2}\, dy + H(0; n+1)\underline{v} \\
&= g(v; n+1) + H(0; n+1)\underline{v},
\end{aligned}
$$

where (45) follows by SOSD. Rearranging extremes of the previous system, we have for all $v > \underline{v}$

$$
\begin{aligned}
g(v; n) &\geq g(v; n+1) + \big[H(0; n+1) - H(0; n)\big]\underline{v} \\
&\geq g(v; n+1), \qquad (46)
\end{aligned}
$$

where the second inequality follows from Step 4 ($H(0; n+1) > H(0; n)$) and $\underline{v} \geq 0$. Now (46) establishes the desired result, that an individual's effort decreases pointwise with the number of team members.

Proof of part 2(b). **Step 1: Rewrite the quantity of interest using the change of variable $y = F(x)$ and a density $\phi(y;n)$.** As derived in the proof of part 1(c) above, the expected best effort of a given team equals

$$
\int_{\underline{v}}^{\bar{v}} g(v) \frac{d}{dv} (F(v))^n\, dv = \int_0^1 F^{-1}(y)\, \phi(y; n)\, dy, \qquad (47)
$$

where $\phi(y;n) = n(N-1)(1-y^n)y^{Nn-2}$.

Step 2: Verify ϕ*(y;n)* is indeed a density and define its cdf. Given N, define the cumulative distribution H on $[0,1]$ by

$$H(y;n) = \left(1 - \frac{(N-1)n^2}{(Nn-1)((N+1)n-1)}\right) + \int_0^y n(N-1)(1-x^n)x^{Nn-2}\,dx. \tag{48}$$

Step 3: Show the average *y* under *H(y; n)* is independent of *n*. We have

$$\int_0^1 y\,dH(y;n) = \frac{N-1}{N(N+1)}.$$

Step 4: Establish *H(y; n+1)–H(y; n)* starts strictly positive, ends at zero, and has only one sign change for *y*$\in [\mathbf{0},\mathbf{1}]$. Observe that

$$\frac{\partial H(0;n)}{\partial n} = \frac{n(N-1)(2Nn+n-2)}{(nN-1)^2(Nn+n-1)^2} > 0$$

so $H(0;n) < H(0;n+1)$. Next we have

$$\frac{d}{dy}\left(H(y;n+1) - H(y;n)\right) = \underbrace{\left(\frac{n+1}{n}y^N - \frac{1-y^n}{1-y^{n+1}}\right)}_{\equiv \Delta(y)} n(N-1)(1-y^{n+1})y^{Nn-2}.$$

Note that $\Delta(0) = -1$, $\Delta(1) = \frac{n+1}{n} - \frac{n}{n+1} > 0$, and $\Delta' > 0$ (cf. footnote 26). Thus, there exists a unique $y^a(n) \in (0,1)$ such that

$$\frac{d}{dy}\Big(H(y\,;\,n+1) - H(y\,;\,n)\Big) \lesseqgtr 0 \iff y \lesseqgtr y^a(n).$$

As in Step 4 of part 2(a), this last condition together with $H(0;n+1) > H(0;n)$ and $H(1;n+1) = H(1;n)$ implies there exists $y^c(n) \in (0,1)$ such that

$$H(y\,;\,n+1) - H(y\,;\,n) \gtreqless 0 \iff y \lesseqgtr y^c(n). \tag{49}$$

Step 5: Show that *H*($\cdot$;*n*) second-order stochastically dominates *H*($\cdot$;*n+1*). We show

$$\int_0^y H(x;n+1)\,dx - \int_0^y H(x;n)\,dx > 0 \qquad \forall y \in (0,1). \tag{50}$$

The validity of (50) follows immediately from (49) for any $y \in (0, y^c(n)]$. So now consider $y \in (y^c(n), 1)$. Because the difference of integrals in (50) is strictly decreasing over this interval (by (49)), we see for any

$y \in (y^c(n), 1)$ that

$$\int_0^y H(x; n+1)\, dx - \int_0^y H(x; n)\, dx \geq \int_0^1 H(x; n+1)\, dx - \int_0^1 H(x; n)\, dx$$
$$= \int_0^1 (1 - H(x; n))\, dx - \int_0^1 (1 - H(x; n+1))\, dx$$
$$= \mathrm{E}[\, y \,|\, n\,] - \mathrm{E}[\, y \,|\, n+1\,] \,|| \qquad > 0,$$

where the final inequality follows from Step 3.

Step 6: Conclude the proof by using SOSD when F^{-1} is assumed concave (i.e., F is assumed convex). Given that F^{-1} is concave (i.e., F is convex), we have

$$\begin{aligned}\mathrm{E}[\, g(v) \,|\, n\,] + H(0; n)\underline{v} &= \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; n)\,\big] \\ &\geq \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; n+1)\,\big] \qquad (51) \\ &= \mathrm{E}[\, g(v) \,|\, n+1\,] + H(0; n+1)\underline{v},\end{aligned}$$

where (51) follows because $H(\,\cdot\,; n)$ SOSD $H(\,\cdot\,; n+1)$. Rearranging extremes of the previous system, we have

$$\begin{aligned}\mathrm{E}[\, g(v) \,|\, n\,] &\geq \mathrm{E}[\, g(v) \,|\, n+1\,] + \big(H(0; n+1) - H(0; n)\big)\underline{v} \\ &\geq \mathrm{E}[\, g(v) \,|\, n+1\,]\,, \qquad (52)\end{aligned}$$

where the second inequality follows from Step 4 Now (52) establishes the desired result, that a team's expected best shot decreases with the number of team members.

Proof of part 2(c). **Step 1: Rewrite the quantity of interest using the change of variable $y = F(x)$ and a density $\phi(y;n)$.** As in the proof of part 1(b), the *contest's* overall expected best shot is

$$\int_{\underline{v}}^{\bar{v}} g(v) \frac{d}{dv} (F(v))^{Nn}\, dv = \int_0^1 F^{-1}(y) \phi(y; n)\, dy.$$

where $\phi(y; n) = n(N-1)(1 - y^{Nn}) y^{Nn-2}$.

Step 2: Verify $\phi(y;n)$ is indeed a density and define its cdf. Integration of ϕ yields

$$\int_0^1 \phi(y; n)\, dy = \frac{n^2 N(N-1)}{(Nn-1)(2nN-1)} < 1,$$

leading us to define the cdf H on $[0, 1]$ by

$$H(y;n) = \left(1 - \frac{n^2 N(N-1)}{(Nn-1)(2nN-1)}\right) + n(N-1)\int_0^y (1 - x^{Nn})x^{Nn-2}\,dx. \tag{53}$$

Step 3: Show the average *y* under *H(y; n)* is independent of *n*. We note that, integrating with respect to y,

$$\mathrm{E}[\,y\,|\,n\,] \equiv \int_0^1 y\,dH(y;n) = \frac{N-1}{2N}. \tag{54}$$

Step 4: Establish *H(y; n+1)–H(y; n)* starts strictly positive, ends at zero, and has only one sign change for *y*$\in [\mathbf{0}, \mathbf{1}]$. We have

$$\frac{\partial H(0;n)}{\partial n} = \frac{nN(N-1)(3nN-2)}{(nN-1)^2(2nN-1)^2} > 0,$$

so

$$H(0;n+1) - H(0;n) > 0. \tag{55}$$

Next, observe that

$$\begin{aligned}\frac{d}{dy}\Big(H(y\,;\,n+1) - H(y\,;\,n)\Big) &= \frac{N-1}{y^2}\left[(n+1)y^{N(n+1)} - (n+1)y^{2N(n+1)} - ny^{nN} + ny^{2nN}\right] \\ &= n(N-1)y^{nN-2}\left(1 - y^{N(n+1)}\right)\left(\frac{n+1}{n}y^N - \frac{1-y^{nN}}{1-y^{N(n+1)}}\right).\end{aligned} \tag{56}$$

As in Step 4 of part 2(a), this last condition together with (55) and $H(1;n+1) - H(1;n) = 0$ implies there exists $y^c(n) \in (0, 1)$ such that

$$H(y\,;\,n+1) - H(y\,;\,n) \gtreqless 0 \iff y \lesseqgtr y^c(n). \tag{57}$$

Step 5: Show that *H*($\cdot$;*n*) second-order stochastically dominates *H*($\cdot$;*n+1*). The conclusion that

$$\int_0^y H(x;n+1)\,dx - \int_0^y H(x;n)\,dx > 0 \qquad \forall y \in (0,1), \tag{58}$$

follows exactly as in Step 5 of part 2(a).

Step 6: Conclude the proof by using SOSD when $\boldsymbol{F^{-1}}$ is assumed concave (i.e., *F* is assumed

convex). Given that F^{-1} is concave (i.e., F is convex), we have

$$\int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^{Nn}\, dv + H(0;n)\underline{v} \geq \int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^{N(n+1)}\, dv + H(0;n+1)\underline{v},$$

just as in Step 6 of part 1 (a). Rearranging extremes of the previous system, we have for all $v > \underline{v}$

$$\begin{aligned}\int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^{Nn}\, dv &\geq \int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^{N(n+1)}\, dv + \big[H(0;n+1) - H(0;n)\big]\underline{v} \\ &\geq \int_{\underline{v}}^{\bar{v}} g(v)\frac{d}{dv}(F(v))^{N(n+1)}\, dv, \end{aligned} \tag{59}$$

where the second inequality follows from Step 4 and $\underline{v} \geq 0$. Now (59) establishes the desired result, that the contest's overall expected best effort decreases with the number of members per team.

Proof of part 2(d). Using the utility calculation given in (26), we have

$$U^*(v;n+1) - U^*(v;n) = U^*(\underline{v}, n+1) - U^*(\underline{v};n) + \int_{\underline{v}}^{v} \left(\int_{n}^{n+1} \frac{\partial b(z;n)}{\partial n}\, dn\right) dz.$$

We now note two facts. First, simple algebra shows that $U^*(\underline{v}, n+1) > U^*(\underline{v}, n)$. Second, $\partial b(v;n)/\partial n$ is positive for low v and negative for large v. Indeed, we have

$$\begin{aligned}\frac{\partial b(v;n)}{\partial n} =& \frac{N-1}{(Nn-1)^2}\left(1 - (F(v))^{nN-1}\right) + \frac{n(N-1)}{nN-1} N (F(v))^{nN-1} \log(F(v)) \\ =& \frac{N-1}{(Nn-1)^2}\left(1 - (F(v))^{nN-1}\right) + \frac{Nn(N-1)}{(Nn-1)^2} (F(v))^{nN-1} \log\Big((F(v))^{nN-1}\Big) \\ =& \frac{N-1}{(Nn-1)^2}\left(1 - (F(v))^{nN-1} + Nn (F(v))^{nN-1} \log\Big((F(v))^{nN-1}\Big)\right). \end{aligned}$$

Letting $x = (F(v))^{nN-1}$, the above has the same sign as does $a(x) = 1 - x + Nn \cdot x \cdot \log(x)$. Since $a(0) = 1$, $a(1) = 0$, and

$$a'(x) = Nn - 1 + \log(x)^{Nn} = \begin{cases} < 0 & \text{if } x < \left(e^{\frac{Nn-1}{Nn}}\right)^{-1} \\ = 0 & \text{if } x = \left(e^{\frac{Nn-1}{Nn}}\right)^{-1} \\ > 0 & \text{if } x > \left(e^{\frac{Nn-1}{Nn}}\right)^{-1} \end{cases},$$

we conclude that $\partial b(v;n)/\partial n$ is strictly positive for low v, then it becomes strictly negative. The two facts just established imply that, if $U^*(\bar{v}; n+1) \geq U^*(\bar{v}; n)$, then $U^*(v; n+1) > U^*(v; n)$ for any $v \in (\underline{v}, \bar{v})$, as we want to show. But $U^*(\bar{v}; n) = \bar{v} - g(\bar{v}; n)$, so part 1(a) of this theorem and continuity of g imply $g(\bar{v}; n) \geq g(\bar{v}; n+1)$, thus concluding the proof. □

Proof of Proposition 2. To establish the comparative statics for each *team's* expected best effort, we follow the proof of part 2(b) of Theorem 1 through Step 5, up to equation (51) in Step 6. For $\underline{v} = 0$, this equation must be modified to reflect that

$$\mathrm{E}[\, g(v) \,|\, n \,] = \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,;\, n) \,\big]\,,$$

where v represents the maximum of n draws from F and $H(\cdot\,;\, n)$ is given by (48). The proposition now follows from the fact, established in Step 5 of the proof of part 2(b) of Theorem 1, that $H(\cdot\,; n+1)$ is a mean-preserving spread of $H(\cdot\,; n)$.

To establish the comparative statics for the *contest's* expected best effort, we follow the proof of part 2(c) of Theorem 1 through Step 5. Given $\underline{v} = 0$, we have

$$\mathrm{E}[\, g(v) \,|\, n \,] = \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,;\, n) \,\big]\,,$$

where v represents the maximum of nN draws from F and $H(\cdot\,;\, n)$ is given by (53). The proposition now follows from the fact, established in Step 5 of the proof of part 2(c) of Theorem 1, that $H(\cdot\,; n+1)$ is a mean-preserving spread of $H(\cdot\,; n)$. $\square$

Proof of Lemma 2. Fix $N \geq 2$. Because

$$\mathrm{corr}\{v, p(v; n)\} = \mathrm{E}[\, vp(v; n) \,] - \mathrm{E}[\, v \,]\, \mathrm{E}[\, p(v; n) \,] = \mathrm{E}[\, vp(v; n) \,] - \frac{1}{N}\mathrm{E}[\, v \,]\,,$$

it suffices to show that $\mathrm{E}[\, vp(v; n) \,]$ decreases with n.

Step 1: Rewrite the quantity of interest using the change of variable *y* = *F(x)* and a density ϕ*(y;n)*. We obtain

$$\begin{aligned}
\mathrm{E}[\, vp(v; n) \,] &= \int_{\underline{v}}^{\bar{v}} v \left(\frac{n-1}{nN-1} + \frac{n(N-1)}{nN-1} \left(F\left(v\right)\right)^{nN-1} \right) f(v)\, dv \\
&= \int_0^1 F^{-1}(y)\, \phi(y; N)\, dy,
\end{aligned}$$

where the first equality uses (1) and the second uses the change of variable $y = F(x)$ and the definition

$$\phi(y; n) \equiv \frac{n - 1 + n(N-1) y^{nN-1}}{nN-1}.$$

The method of proof will be to use first-order stochastic dominance (FOSD), regarding $F^{-1}(y)$ as a "utility function" and $\phi(y;n)$ as a density.

Step 2: Verify ϕ(y;n) is indeed a density and define its cdf. Integration of ϕ over $[0,1]$ yields

$$\int_0^1 \phi(y;n)\,dy = \frac{1}{N} < 1,$$

leading us to define the cdf H on $[0,1]$ by

$$H(y;n) = \left(1 - \frac{1}{N}\right) + \int_0^y \frac{n-1+n(N-1)x^{nN-1}}{nN-1}\,dx.$$

Step 3: Show that H(· ; n) FOSD H(· ; n+1). First observe that

$$\begin{aligned}
\frac{d}{dy}\left(H(y;n+1) - H(y;n)\right) &= \phi(y;n+1) - \phi(y;n) \\
&= \frac{(N-1)\left[y + y^{nN}\left(-y^N + n(nN+N-1)\left(-1+y^N\right)\right)\right]}{(nN-1)(N-1+nN)y} \\
&= \frac{(N-1)\left[1 - y^{nN-1}\left(y^N + n(nN+N-1)(1-y^N)\right)\right]}{(nN-1)(nN+N-1)} \\
&= \frac{(N-1)\Delta(y)}{(nN-1)(nN+N-1)}, \qquad (60)
\end{aligned}$$

where $\Delta(y) \equiv 1 - y^{nN-1}\left(y^N + n(nN+N-1)(1-y^N)\right)$. Furthermore,

$$\begin{aligned}
\Delta'(y) &= (nN-1)(nN+N-1)y^{nN-2}\left((n+1)y^N - n\right) \\
&= \begin{cases} < 0 & \text{if } y < \left(\frac{n}{n+1}\right)^{1/N} \\ > 0 & \text{if } y > \left(\frac{n}{n+1}\right)^{1/N}. \end{cases} \qquad (61)
\end{aligned}$$

Because $\Delta(0) = 1$ and $\Delta(1) = 0$, it follows from (61) that there exists $y^a \in \left(0, \left(\frac{n}{n+1}\right)^{1/N}\right)$ such that $\Delta'(y) \gtrless 0$ as $y \lesseqgtr y^a$. From this and (60) it now follows that $H(y;n+1) - H(y;n)$ is strictly quasiconcave in y. Finally, because $H(0;n+1) - H(0;n) = 0$ and $H(1;n+1) - H(1;n) = 0$, it follows that $H(y;n+1) - H(y;n) > 0$ for all $y \in (0,1)$, thus establishing the desired FOSD ordering.

Step 4: Conclude the proof using Step 3. Noting that $\underline{v} = F^{-1}(0)$, we have

$$\begin{aligned}
\mathrm{E}[\,vp(v;n+1)\,] + H(0;n+1)\underline{v} &= \int_0^1 F^{-1}(y)\phi(y;n+1)\,dy + H(0;n+1)\underline{v} \\
&= \mathrm{E}\left[\,F^{-1}(y)\,|\,H(\,\cdot\,;n+1)\,\right]
\end{aligned}$$

$$
\begin{aligned}
&< \mathrm{E}\big[\, F^{-1}(y) \,|\, H(\,\cdot\,; n) \,\big] \\
&= \int_0^1 F^{-1}(y)\phi(y;n)\, dy + H(0;n)\underline{v} \\
&= \mathrm{E}[\, vp(v;n) \,] + H(0;n)\underline{v},
\end{aligned}
$$

where the inequality follows from strict FOSD (Step 3) since F^{-1} is increasing. Because $H(0;n{+}1) = H(0;n)$, it now follows that $\mathrm{E}[\, vp(v;n+1) \,] < \mathrm{E}[\, vp(v;n) \,]$, as was to be shown. □

Proof of Proposition 3. Proof of part 1. From (10) we see the cdf of the overall best shot is

$$
H_C^{BS}(\gamma;N) = \left(\frac{Nnt+1-t}{(N-1)nt}\,\gamma \right)^{\frac{Nnt}{Nnt+1-t}},
$$

which, as $N \to \infty$, clearly converges to γ, the uniform distribution on $[0,1]$.

Next, define $z = \frac{Nnt}{Nnt+1-t}$ so we can write

$$
\log\big(\, H_C^{BS}(\gamma;N) \,\big) = z \log\left(\frac{N}{z(N-1)}\gamma \right).
$$

If $t \leq 1$, then

$$
\frac{\partial \log\big(\, H_C^{BS}(\gamma;N) \,\big)}{\partial N} = -\frac{z}{N(N-1)} + \left(\log\left(\frac{N}{z(N-1)}\gamma \right) - 1 \right) \frac{\partial z}{\partial N} < 0
$$

because $\partial z / \partial N \geq 0$ when $t \leq 1$. Thus, when $t \leq 1$, $H_C^{BS}(\gamma;N)$ decreases pointwise with N for $\gamma > 0$.

Proof of part 2. From (10) it is immediate that if $t = 1$, then a team's best-shot cdf is simply

$$
H^{BS}(\gamma\,;n) = \left(\frac{N}{N-1}\gamma \right)^{\frac{1}{N-1}}, \tag{62}
$$

which is obviously independent of n. Now consider $t \neq 1$. Fix t and γ; define $z = nt/(Nnt+1-t)$. Then we rewrite (10) as

$$
H^{BS}(\gamma\,;n) = \left(\frac{Nnt+1-t}{nt} \times \frac{nt}{(N-1)nt}\,\gamma \right)^{\frac{nt}{Nnt+1-t}} = \left(\frac{\gamma}{z(N-1)} \right)^{z},
$$

so

$$
\log\big(\, H^{BS} \,\big) = z \left(\log\left(\frac{\gamma}{N-1} \right) - \log(\, z \,) \right).
$$

Now

$$
\frac{\partial}{\partial n} \log\big(\, H^{BS} \,\big) = \frac{\partial \log\big(\, H^{BS} \,\big)}{\partial z} \times \frac{\partial z}{\partial n}
$$

$$= \underbrace{\left(\log\left(\frac{\gamma}{z(N-1)}\right) - 1\right)}_{<0 \text{ because } \gamma<z(N-1)} \times \frac{t(1-t)}{(Nnt+1-t)^2}.$$

Now suppose, $t < 1$. The previous calculation shows that $\log\left(H^{BS}\right)$ is decreasing in n, so H^{BS} is decreasing in n; that is, as n increases, H^{BS} shifts rightward in the sense of first-order stochastic dominance. The reverse conclusion holds when $t > 1$. Even for $t \neq 1$, it is easily seen from (10) that a team's best-shot cdf has limiting distribution $\left(\frac{N\gamma}{N-1}\right)^{1/N}$ as $n \to \infty$. $\square$

Proof of Proposition 4. Part 1. Using (9), we have

$$\mathrm{E}[\, g \,|\, n \,] = \int_0^1 g(v) t v^{t-1} \, dv = \frac{nt^2}{(2nt+1)(2nt+1-t)}.$$

Observe that

$$\frac{\partial}{\partial n} \mathrm{E}[\, g \,|\, n \,] = \frac{t^2(1-t-4n^2t^2)}{(2nt+1-t)^2(2nt+1)^2}. \tag{63}$$

Examining the numerator of (63), we see $\mathrm{E}[\, g \,|\, n \,]$ is strictly quasiconcave in n. Therefore, if at t_0 it is the case that $\mathrm{E}[\, g \,|\, n=1 \,] = \mathrm{E}[\, g \,|\, n=2 \,]$, then for $t = t_0$ we have that $\mathrm{E}[\, g \,|\, n \,]$ is strictly decreasing in n for $n \geq 2$. Moreover, because $\partial(1-t-4n^2t^2)/\partial t < 0$, it follows that $\mathrm{E}[\, g \,|\, n \,]$ is strictly decreasing in n for all $t > t_0$. It is easily verified that $\mathrm{E}[\, g \,|\, n=1 \,] = \mathrm{E}[\, g \,|\, n=2 \,]$ for $t = (\sqrt{33}-1)/16 \approx 0.296353$. Therefore, if $t > 0.296535$, then $\mathrm{E}[\, g \,|\, n \,]$ is strictly decreasing in n, as we wanted to show. Given quasiconcavity of $\mathrm{E}[\, g \,|\, n \,]$, for $t < (\sqrt{33}-1)/16$ we have that $\mathrm{E}[\, g \,|\, n \,]$ initially increases with n and then decreases in n, thus completing the proof of Part 1.

Part 2. Multiplying individual efforts by n, we see a team's expected effort equals

$$\mathrm{E}[\, ng \,|\, n \,] = \frac{n^2t^2}{(2nt+1)(2nt+1-t)},$$

from which we obtain

$$\frac{\partial}{\partial n} \mathrm{E}[\, ng \,|\, n \,] = \frac{2nt^2\big[1-t+nt(2-t)\big]}{(2nt+1-t)^2(2nt+1)^2}. \tag{64}$$

From (64) it follows that if $t \leq 1$, then $\mathrm{E}[\, ng \,|\, n \,]$ is strictly increasing in n; and if $t \geq 2$, then $\mathrm{E}[\, ng \,|\, n \,]$ is strictly decreasing in n. So now consider $t \in (1,2)$.

For $t \in (1,2)$, (64) shows $\mathrm{E}[\, ng \,|\, n \,]$ is quasiconvex in n. We see that $\mathrm{E}[\, ng \,|\, n=1 \,] = \mathrm{E}[\, ng \,|\, n=2 \,]$ if and only if $t = (5+\sqrt{73})/8 \approx 1.693$. Because $\partial[1-t+2n(2-t)]/\partial t = -1-2n(t-1) < 0$ for all $t > 1$, it now follows that $\mathrm{E}[\, ng \,|\, n \,]$ is strictly increasing for $t < 1.693$. And for $t \in (1.693, 2)$, $\mathrm{E}[\, ng \,|\, n \,]$ is initially decreasing in n but eventually increases.

Finally, we see

$$\mathrm{E}[\,ng\,|\,n\,] = \frac{t^2}{\left(2t + \dfrac{1}{n}\right)\left(2t + \dfrac{1-t}{n}\right)} \to \frac{1}{4} \qquad \text{as } n \to \infty.$$

□

Proof of Proposition 5. Using (7) and (9), and then taking expectations, we have

$$\mathrm{E}[\,U^*(v)\,|\,n\,] = \frac{t\left(1 + (2n-1)t + (n-1)nNt^2\right)}{(1+t)(1+nNt)(1+(nN-1)t)}.$$

The proof now follows by setting $N = 2$ and differentiating the above and recognizing that $\mathrm{E}[\,U^*(v)\,|\,n\,]$ need only be compared at integer values of n, as in the proof of Proposition 4. We omit these details. □

Proof of Proposition 6. Follow the procedure of Amann and Leininger (1996). Define the "value matching" function $k(v_1) = \varphi_2(g_1(v_1))$. Note that in equilibrium k must satisfy the initial condition $k(\bar{v}) = \bar{v}$. Also, we have $k'(v_1) = \varphi_2'(g_1(v_1)) \cdot g_1'(v_1)$. Now evaluate the system (11)–(12) at $\gamma = g_1(v_1)$ to get

$$c_1 = nv_1\big(F(v_1)\big)^{m-1}\big(F(k(v_1))\big)^{n-1} f(k(v_1))\,\varphi_2'(g_1(v_1)); \tag{65}$$

and

$$c_2 = mk(v_1)\big(F(v_1)\big)^{m-1}\big(F(k(v_1))\big)^{n-1} f(v_1)\,\varphi_1'(g_1(v_1)). \tag{66}$$

We know that $\varphi_1'(\gamma) = 1/g_1'(\varphi_1(\gamma))$, so, again evaluating at $\gamma = g_1(v_1)$, we obtain $\varphi_1'(g_1(v_1)) = 1/g_1'(v_1)$, which in (66) yields

$$g_1'(v_1)c_2 = mk(v_1)\big(F(v_1)\big)^{m-1}\big(F(k(v_1))\big)^{n-1} f(v_1). \tag{67}$$

Now multiply both sides of (65) by $g_1'(v_1)$ and use (67) to simplify common terms and obtain[27]

$$\frac{c_1\,m\,k(v_1)\,f(v_1)}{c_2\,n\,v_1\,f(k(v_1))} = \varphi_2'(g_1(v_1)) \cdot g_1'(v_1) = k'(v_1). \tag{68}$$

Now (68) forms a standard first-order differential equation for $k(\cdot)$, with initial condition $k(\bar{v}) = \bar{v}$. The conditions on f and f', guarantee the solution $k(v_1)$ exists and is unique (see Theorem 1, p. 162, of Hirsch and Smale, 1974).

Note that, if $c_1\,m = c_2\,n$, then the solution to (68) is $k(v) = v$, implying $g_1 = g_2$. If $c_1\,m < c_2\,n$ instead, we obtain

$$k(v) > v, \quad \forall v \in (\underline{v}, \bar{v}). \tag{69}$$

[27]The differential equation (68) is essentially the same as (1) of Amann and Leininger (1996).

To see this, note first that (69) must hold in a right-neighborhood of $\bar{v}$, since, by (68), $k(\bar{v}) = \bar{v}$ but $k'(\bar{v}) < 1$. Suppose now (69) fails. By continuity of k, then there exists a point v' such that $k(v') = v'$. But note that at any such point (68) implies $k'(v') < 1$. Therefore, it is impossible for (69) to hold in a right-neighborhood of $\bar{v}$, and we have reached a contradiction.

Then, given the unique solution k, the contribution of the members of the first group is uniquely determined by integrating (67) and using the boundary condition $g_1(\underline{v}) = 0$:

$$g_1(v_1) = \int_{\underline{v}}^{v_1} \frac{mk(y)\big(F(y)\big)^{m-1}\big(F(k(y))\big)^{n-1}f(y)}{c_2}\,dy;$$

correspondingly, $g_2(v_2) = g_1(k^{-1}(v_2))$ for $v_2 \geq k(\underline{v})$ and $g_2(v_2) = 0$ for $v_2 < k(\underline{v})$. □

Proof of Proposition 7. We use the definition of k and equation (69) derived in the proof of Proposition 6. By contradiction, suppose there exists some $v' \in (\underline{v}, \bar{v})$ such that $g_1(v') \leq g_2(v')$. Since φ_2 is increasing, we then immediately contradict (69): $k(v') = \varphi_2(g_1(v')) \leq \varphi_2(g_2(v')) = v'$. □

Derivation of Equilibrium Strategies in Table 1. Case 1. $m = 1$, $n = 2$, and $F(v) = \sqrt{v}$. From (18) we obtain

$$2\sqrt{\varphi_1} = \sqrt{\varphi_1\varphi_2} + \sqrt{\varphi_2},$$

which can be rewritten as

$$\sqrt{\varphi_2} = \frac{2\sqrt{\varphi_1}}{1+\sqrt{\varphi_1}} \tag{70}$$

and

$$\sqrt{\varphi_1} = \frac{\sqrt{\varphi_2}}{2-\sqrt{\varphi_2}}. \tag{71}$$

From (12) we obtain

$$\begin{aligned}
1 &= \varphi_2\varphi_2^{1/2}\,\frac{1}{2}\varphi_1^{-1/2}\varphi_1' \\
&= \frac{1}{2}\varphi_2^{3/2}\varphi_1^{-1/2}\varphi_1' \\
&= \frac{1}{2}\left(\frac{2\sqrt{\varphi_1}}{1+\sqrt{\varphi_1}}\right)^3 \varphi_1^{-1/2}\varphi_1' && \text{(by (70))} \\
&= \frac{4\varphi_1}{(1+\sqrt{\varphi_1})^3}\,\varphi_1'.
\end{aligned}$$

Integration of the last expression, yields

$$\gamma + A = 4\left[2\sqrt{\varphi_1} - \frac{6}{1+\sqrt{\varphi_1}} + \frac{1}{(1+\sqrt{\varphi_1})^2} - 6\log(\,1+\sqrt{\varphi_1}\,)\right],$$

for some constant of integration A. The boundary condition $\varphi_1(0) = 0$ implies $A = -20$, from which, recalling $\gamma = g_1(v)$ and $\varphi_1(\gamma) = v$, we deduce

$$g_1(v) = 4\left(2\sqrt{v} - \frac{6}{1+\sqrt{v}} + \frac{1}{(1+\sqrt{v})^2} + 5 - 6\log\big(\,1+\sqrt{v}\,\big)\right).$$

To derive g_2 we proceed analogously using (11):

$$\begin{aligned} 1 &= 2\varphi_1\varphi_2^{1/2}\ \frac{1}{2}\varphi_2^{-1/2}\varphi_2' \\ &= \varphi_1\varphi_2' \\ &= \frac{\varphi_2}{(2-\sqrt{\varphi_2})^2}\varphi_2' && \text{(by (71)).} \end{aligned}$$

Integration of the last expression, yields

$$\gamma + A = 24\log(\,2-\sqrt{\varphi_2}\,) + \frac{16}{2-\sqrt{\varphi_2}} + 8\sqrt{\varphi_2} + \varphi_2,$$

for some constant of integration A. The boundary condition $\varphi_2(0) = 0$ implies $A = 24\log(\,2\,) + 8$, from which we deduce

$$g_2(v) = 24\log\left(\,1 - \frac{\sqrt{v}}{2}\,\right) + \frac{16}{2-\sqrt{v}} + 8\sqrt{v} + v - 8.$$

Case 2. $m = 1$, $n = 2$, and $F(v) = v^2$. From (18) we obtain $\varphi_2 = (1+\varphi_1)/2$. Use this in (12) to obtain

$$\begin{aligned} 1 &= \varphi_2\varphi_2^2\ 2\varphi_1\varphi_1' \\ &= 2\left(\frac{1}{2}(1+\varphi_1)\right)^3 \varphi_1\varphi_1' && \text{(using } \varphi_2 = (1+\varphi_1)/2\text{)} \\ &= \frac{1}{4}\left(\varphi_1 + 3\varphi_1^2 + 3\varphi_1^3 + \varphi_1^4\right)\varphi_1'. \end{aligned}$$

Integration of the last expression, together with the boundary condition $\varphi_1(0) = 0$ yields

$$\gamma = \frac{1}{4}\left(\frac{1}{2}\varphi_1^2 + \varphi_1^3 + \frac{3}{4}\varphi_1^4 + \frac{1}{5}\varphi_1^5\right),$$

which becomes

$$g_1(v) = \frac{1}{4}\left(\frac{1}{2}v^2 + v^3 + \frac{3}{4}v^4 + \frac{1}{5}v^5\right).$$

To derive g_2 we proceed analogously using (11):

$$\begin{aligned} 1 &= 2\varphi_1\varphi_2 \; 2\varphi_2\varphi_2' \\ &= 4(2\varphi_2 - 1)\varphi_2^3\varphi_2' && \text{(using } \varphi_2 = (1+\varphi_1)/2\text{)} \\ &= \left(8\varphi_2^4 - 4\varphi_2^3\right)\varphi_2'. \end{aligned}$$

Integrating the previous expression, with the boundary condition (19) $\varphi_2(0) = 1/2$ yields

$$\gamma = \frac{1}{80} + \frac{8}{5}\varphi_2^5 - \varphi_2^4. \tag{72}$$

Because $\varphi_2(0) = 1/2$, we have $g_2(v) = 0$ for all $v \leq 1/2$. For $v > 1/2$, (72) becomes $g_2(v) = \frac{1}{80} + \frac{8}{5}v^5 - v^4$. $\square$

Proof of Proposition 8. Let $\bar{g}$ denote the maximum effort under the specified equilibrium strategies. Then for any $\gamma \in (0, \bar{g})$ we have

$$H_1^{BS}(\gamma) = \Pr(g_1(v_k) \leq \gamma, k = 1, \ldots, m) = \Pr(v_k \leq \varphi_1(\gamma), k = 1, \ldots, m) = (\varphi_1(\gamma))^{mt}$$

and

$$H_2^{BS}(\gamma) = \Pr(g_2(v_k) \leq \gamma, k = 1, \ldots, n) = \Pr(v_k \leq \varphi_2(\gamma), k = 1, \ldots, n) = (\varphi_2(\gamma))^{nt}$$

From (14) we see that, when $t = 1$, the two groups' best-shot cdfs coincide.

Next suppose $t > 1$. Then, for any $\gamma \in (0, \bar{g})$,

$$\begin{aligned} H_1^{BS}(\gamma) < H_2^{BS}(\gamma) &\iff (\varphi_1(\gamma))^{mt} < (\varphi_2(\gamma))^{nt} \\ &\iff (\varphi_1(\gamma))^{mt} < \left[\left(1 - \frac{m}{n}\right) + \frac{m}{n}(\varphi_1(\gamma))^{t-1}\right]^{\frac{nt}{t-1}} && \text{(using (18))} \\ &\iff (\varphi_1(\gamma))^{\frac{m}{n}(t-1)} < \left(1 - \frac{m}{n}\right) + \frac{m}{n}(\varphi_1(\gamma))^{t-1} && \text{(because } t - 1 > 0\text{)} \\ &\iff z^r - rz + r - 1 < 0, \end{aligned} \tag{73}$$

where $r \equiv m/n$ and $z \equiv (\varphi_1(\gamma))^{t-1}$. Because $r \in (0,1)$, $z^r - rz + r - 1$ is strictly concave in z, reaching a maximum of 0 at $z = 1$, where $\gamma = \bar{g}$. Thus, the inequality in (73) holds, implying $H_1^{BS}(\gamma) < H_2^{BS}(\gamma)$ for all $\gamma \in (0, \bar{g})$.

Lastly, suppose $t < 1$. Then, for any $\gamma \in (0, \bar{g})$,

$$\begin{aligned}
H_1^{BS}(\gamma) > H_2^{BS}(\gamma) &\iff (\varphi_1(\gamma))^{mt} > (\varphi_2(\gamma))^{nt} \\
&\iff (\varphi_1(\gamma))^{mt} > \left[\left(1 - \frac{m}{n}\right) + \frac{m}{n}(\varphi_1(\gamma))^{t-1}\right]^{\frac{nt}{t-1}} && \text{(using (18))} \\
&\iff (\varphi_1(\gamma))^{\frac{m}{n}(t-1)} < \left(1 - \frac{m}{n}\right) + \frac{m}{n}(\varphi_1(\gamma))^{t-1} && \text{(because } t-1<0\text{)} \\
&\iff z^r - rz + r - 1 < 0, && (74)
\end{aligned}$$

where $r \equiv m/n$ and $z \equiv (\varphi_1(\gamma))^{t-1}$. As in the previous case, the inequality in (74) holds, implying $H_1^{BS}(\gamma) > H_2^{BS}(\gamma)$ for all $\gamma \in (0, \bar{g})$. □

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